Design and Analysis of Algorithms

06-02
Paths in Graph

Fastest Route: Dijkstra's Algorithm

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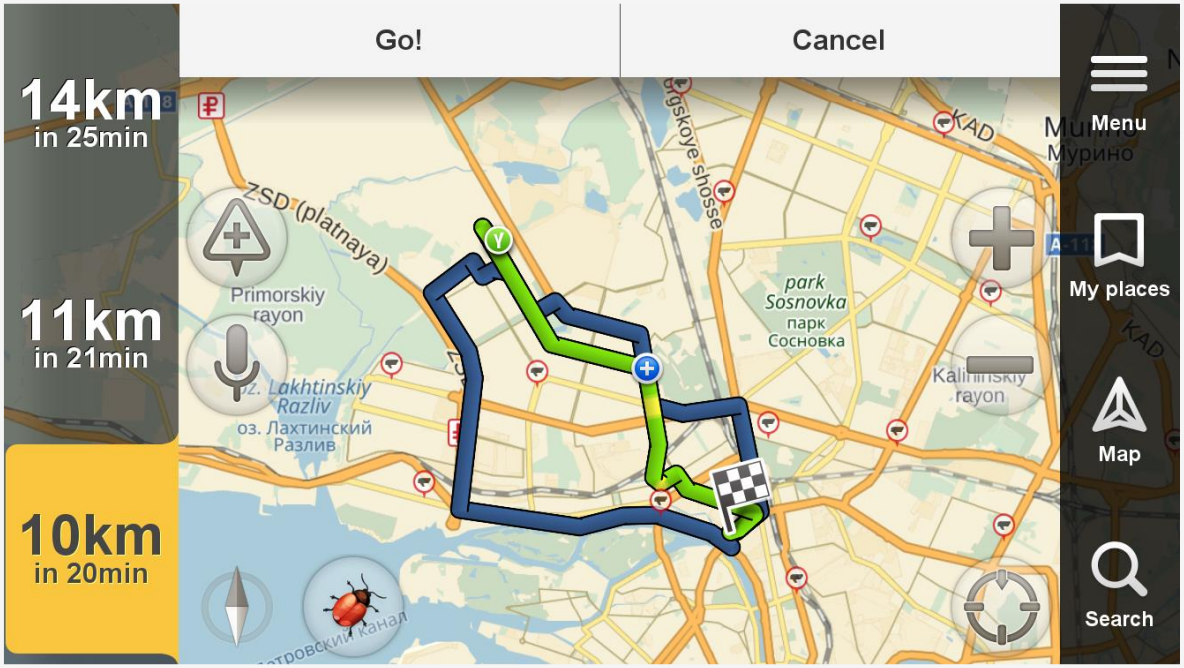
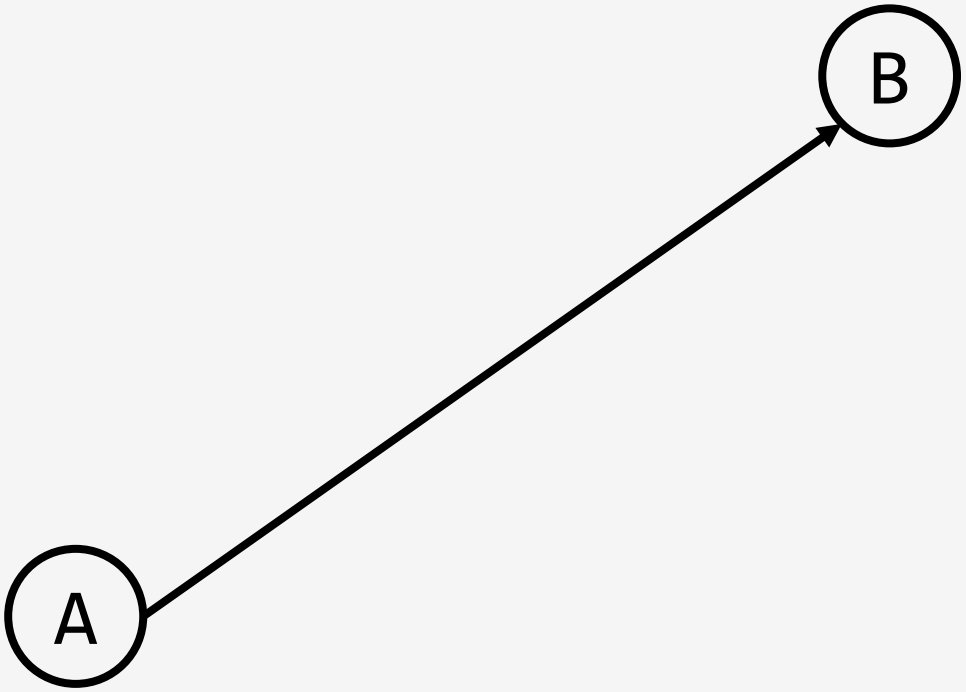
Fastest Route

What is the fastest route to get home from work?



Fastest Route



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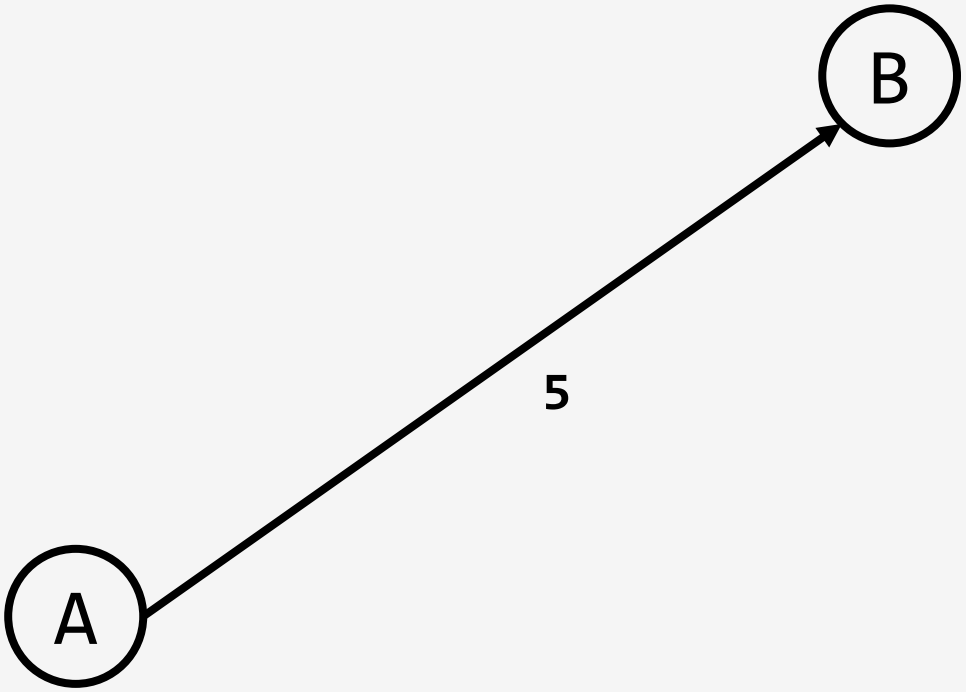

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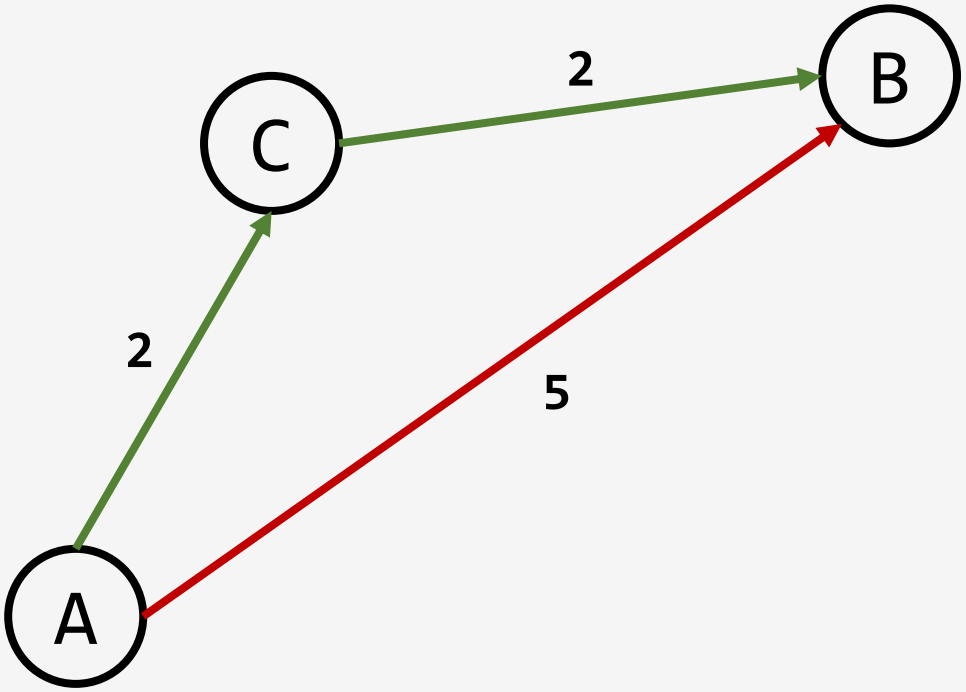

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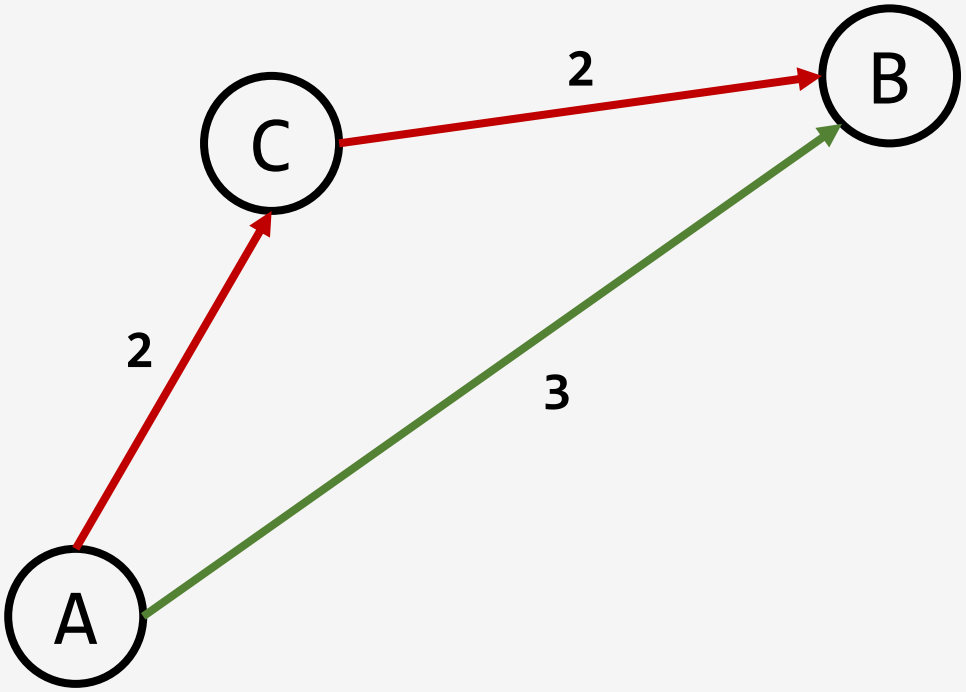
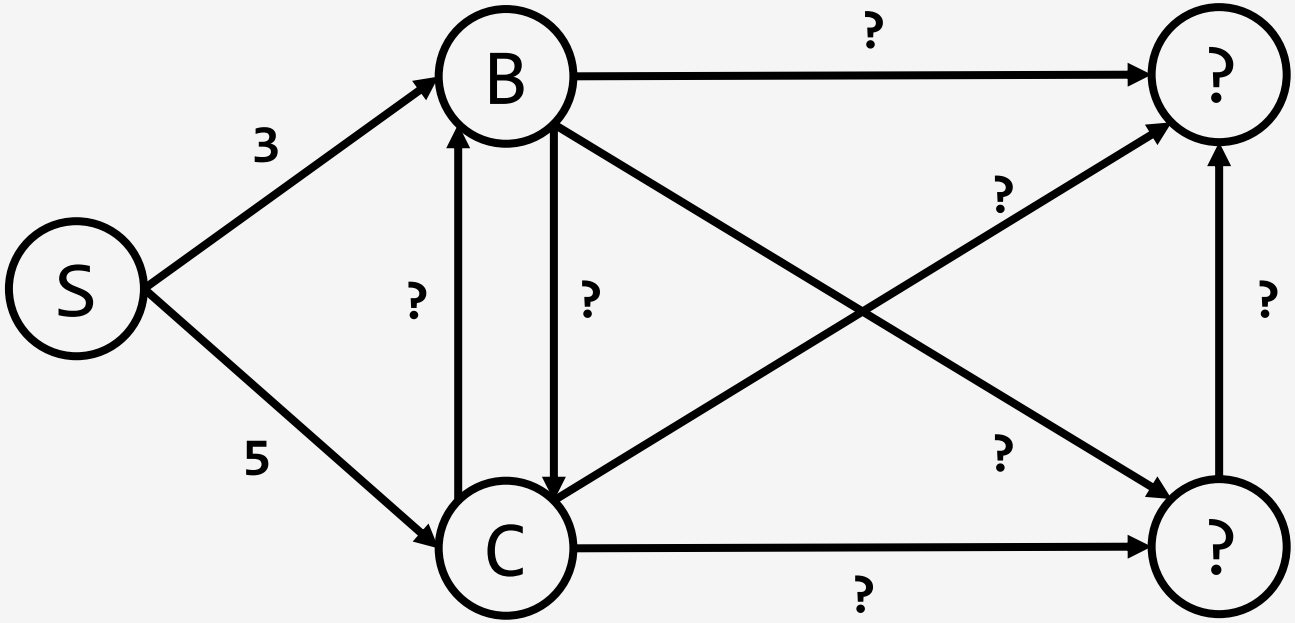


Intuition

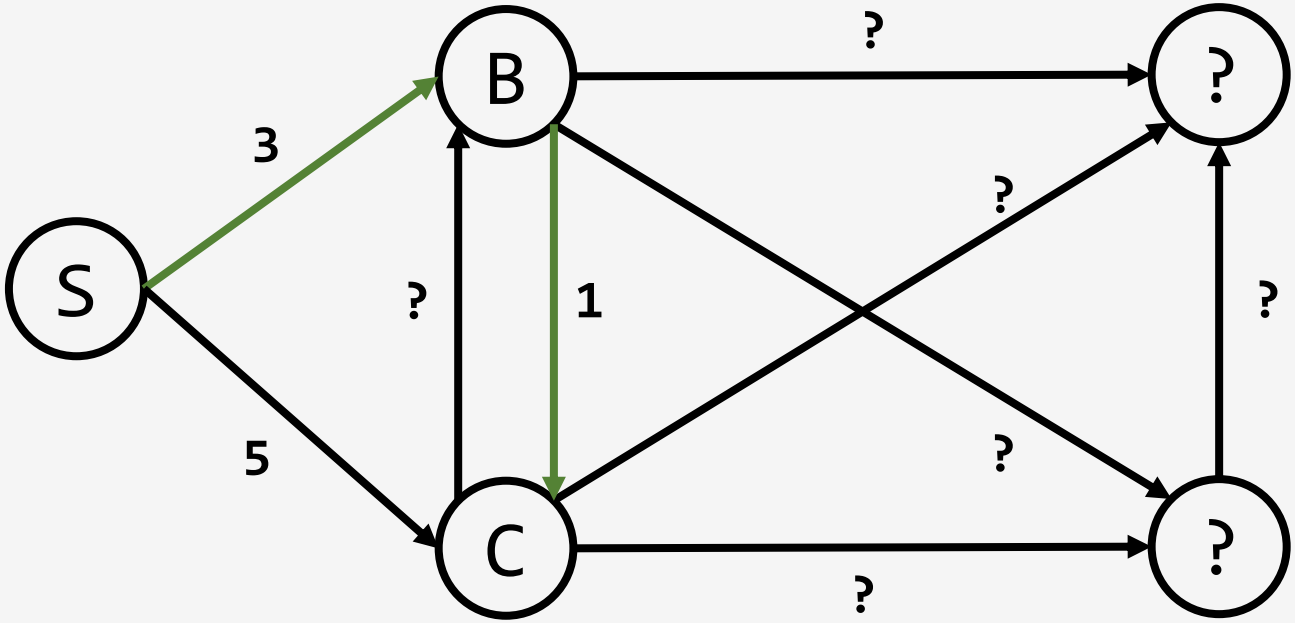
Assume that we stay at S and observe two outgoing edges:



Can we be sure that the distance from S to C is 5?

Intuition

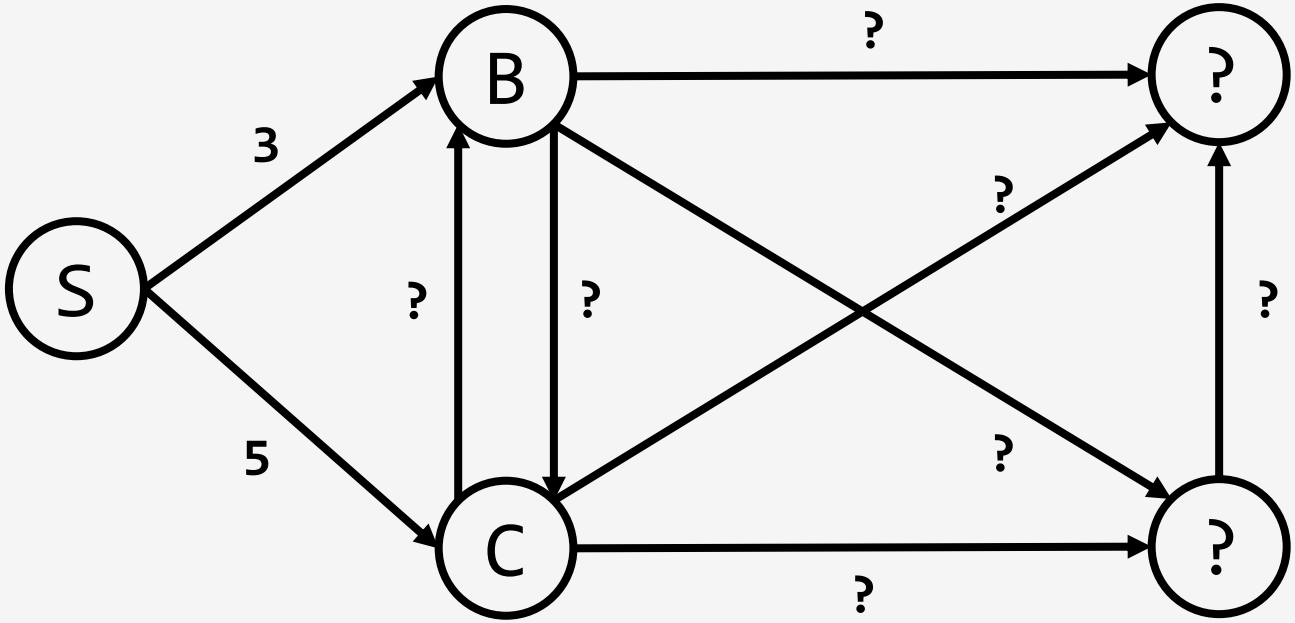
Can we be sure that the distance from S to C is 5?



No, because the weight of the edge (B, C) might be equal to, say, 1.

Intuition

Can we be sure that the distance from S to B is 3?



Yes, because there are no negative weight edges.

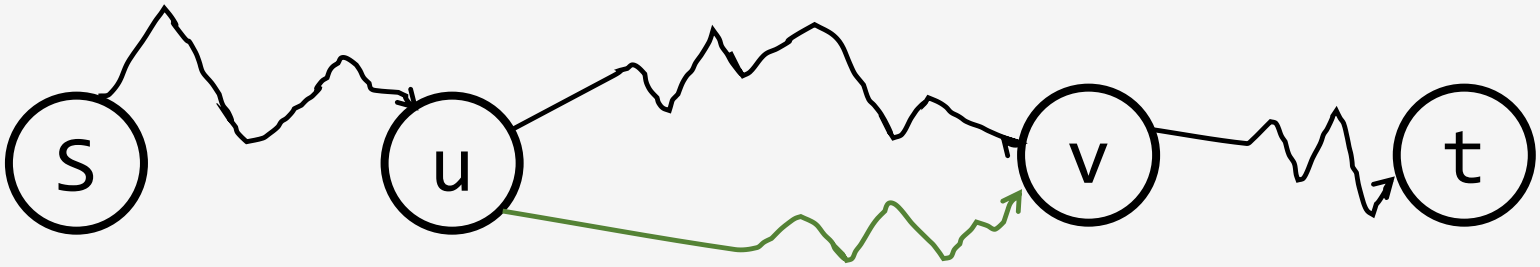
Naïve Algorithm

Optimal substructure

Any sub-path of an optimal path is also optimal.

Proof

Consider an optimal path from S to t and two vertices u and v on this path. If there were a shorter path from u to v, we would get a shorter path from S to t.



Corollary

If $S \rightarrow \ldots \rightarrow u \rightarrow t$ is a shortest path from S to t, then

$$d(S, t) = d(S, u) + w(u, t)$$

Edge Relaxation

dist[v] will be an upper bound on the actual distance from S to v.

The edge relaxation procedure for an edge (u, v) just checks whether going from S to v through u improves the current value of dist[v].

Edge Relaxation

```
Relax((u, v) ∈ E)
        if dist[v] > dist[u] + w(u, v):
            dist[v] ← dist[u] + w(u, v)
            prev [v] ← u
```

Naïve Approach

```
Naïve( G, S)
        for all u ∈ V :
            dist[u] ← ∞
            prev [u] ← nil
        dist[S] ← 0
        do:
            relax all the edges
        while at least one dist changes
```

Correct Distances

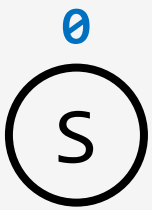
Lemma

After the call to Naïve algorithm all the distances are set correctly.

Dijkstra's Algorithm

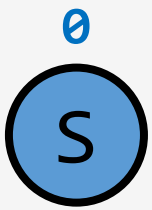
Intuition

initially, we only know the distance to S



Intuition

let's relax all the edges from S



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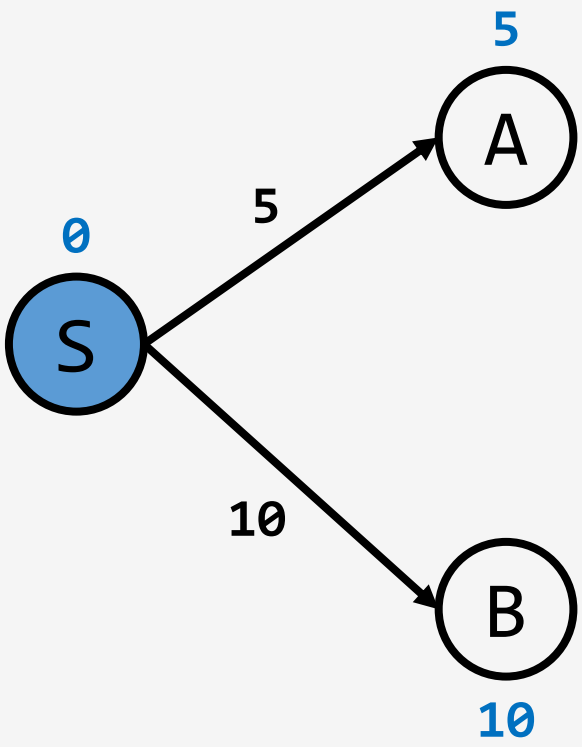
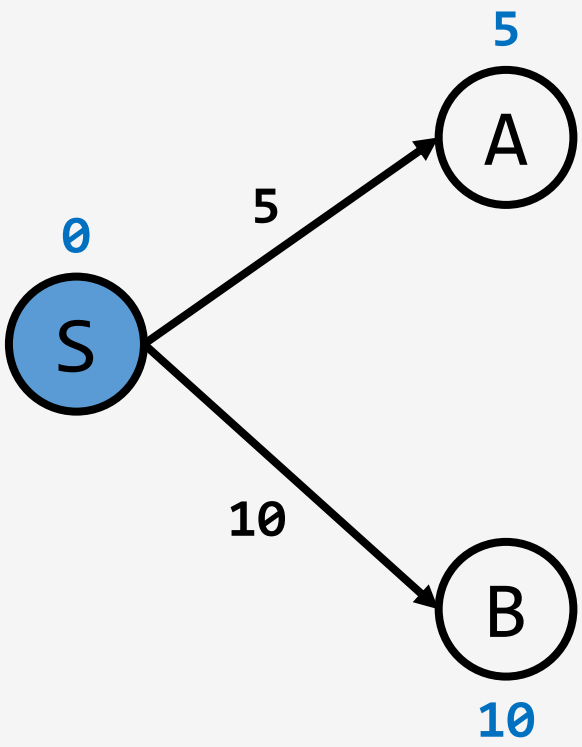


Intuition

we now know the distance for A



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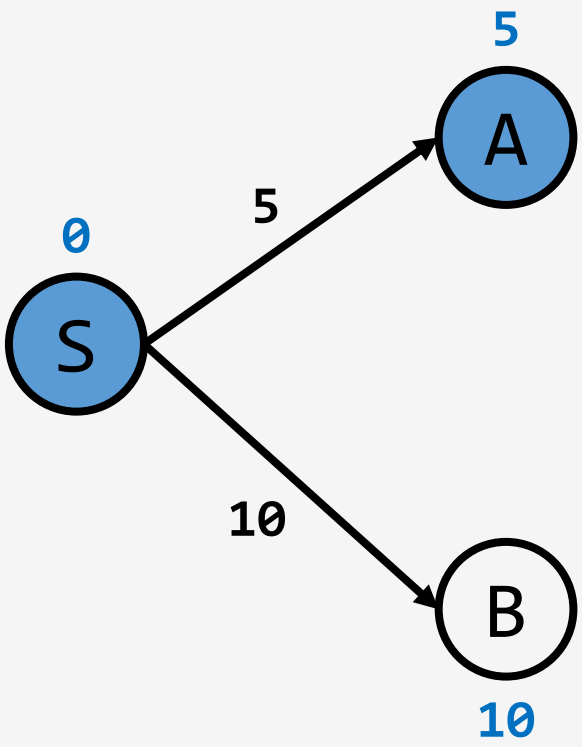
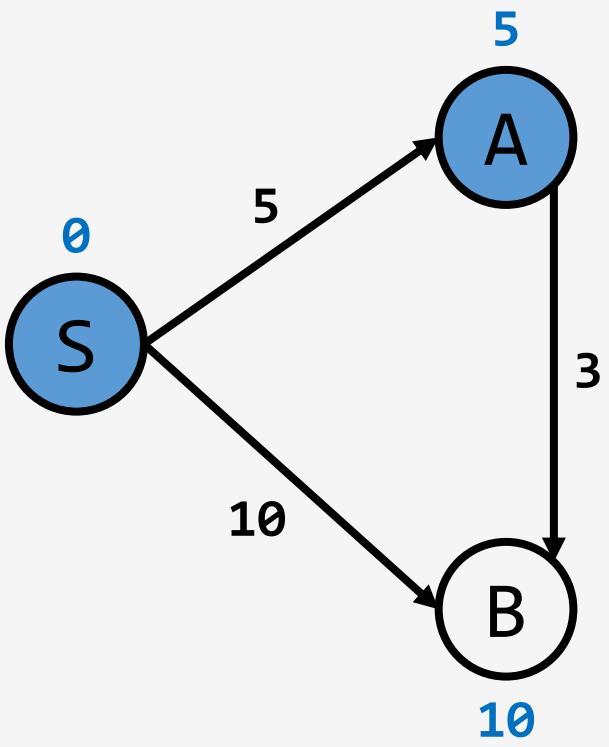


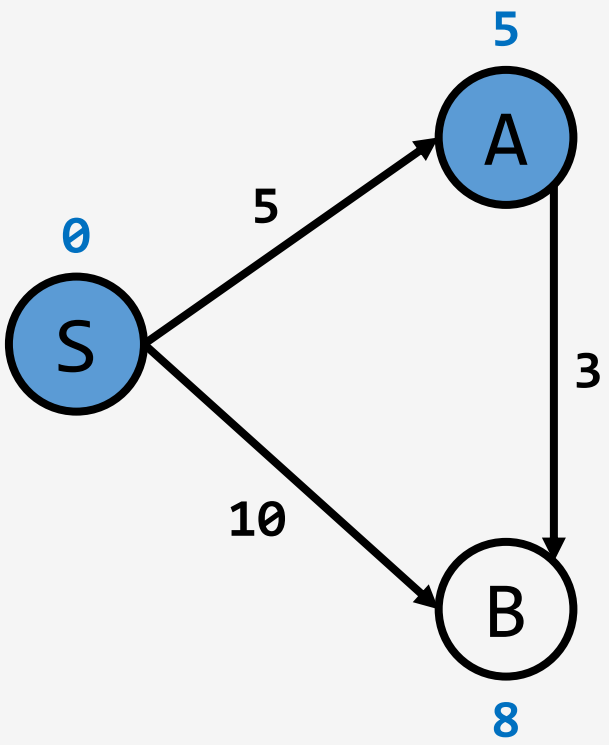
Intuition

now, let's relax all the edges from A



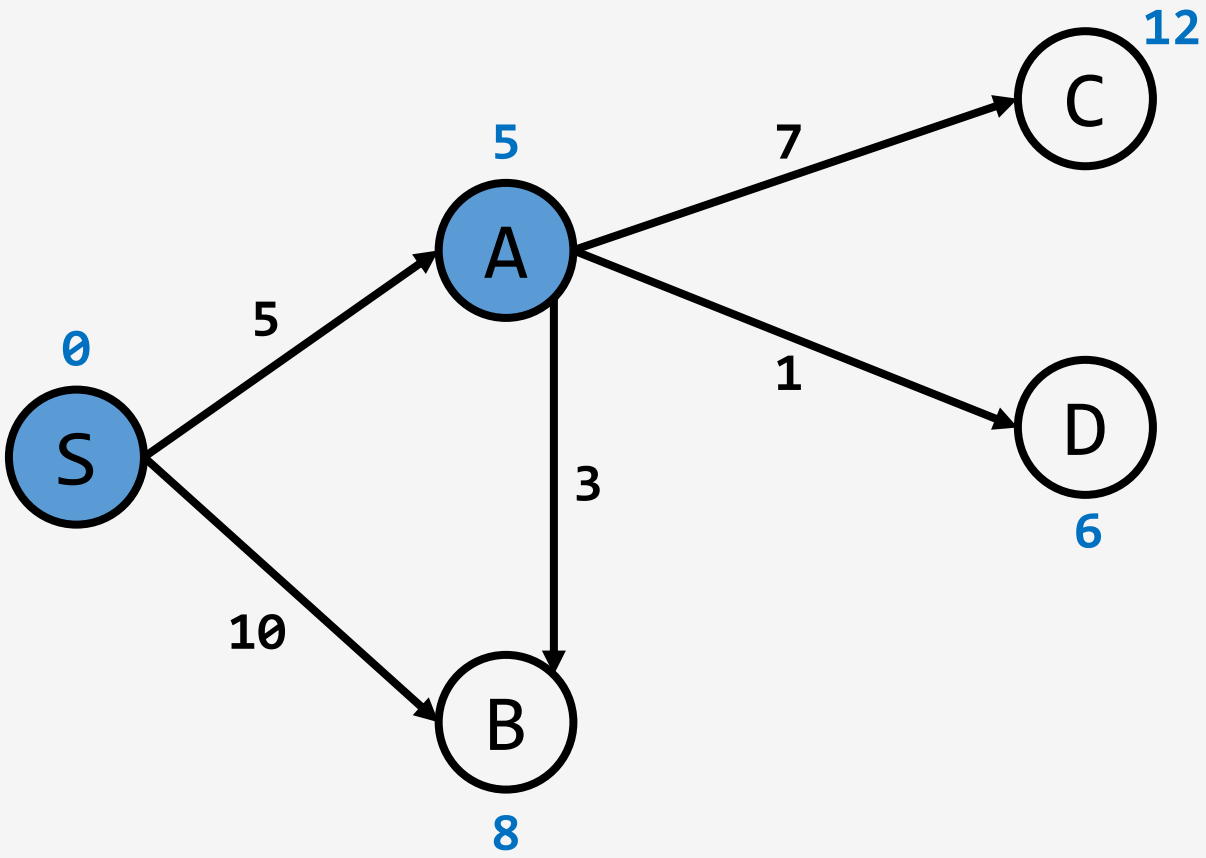
Intuition

we discover an edge (A, B) of weight 3 that updates dist[B]



Intuition

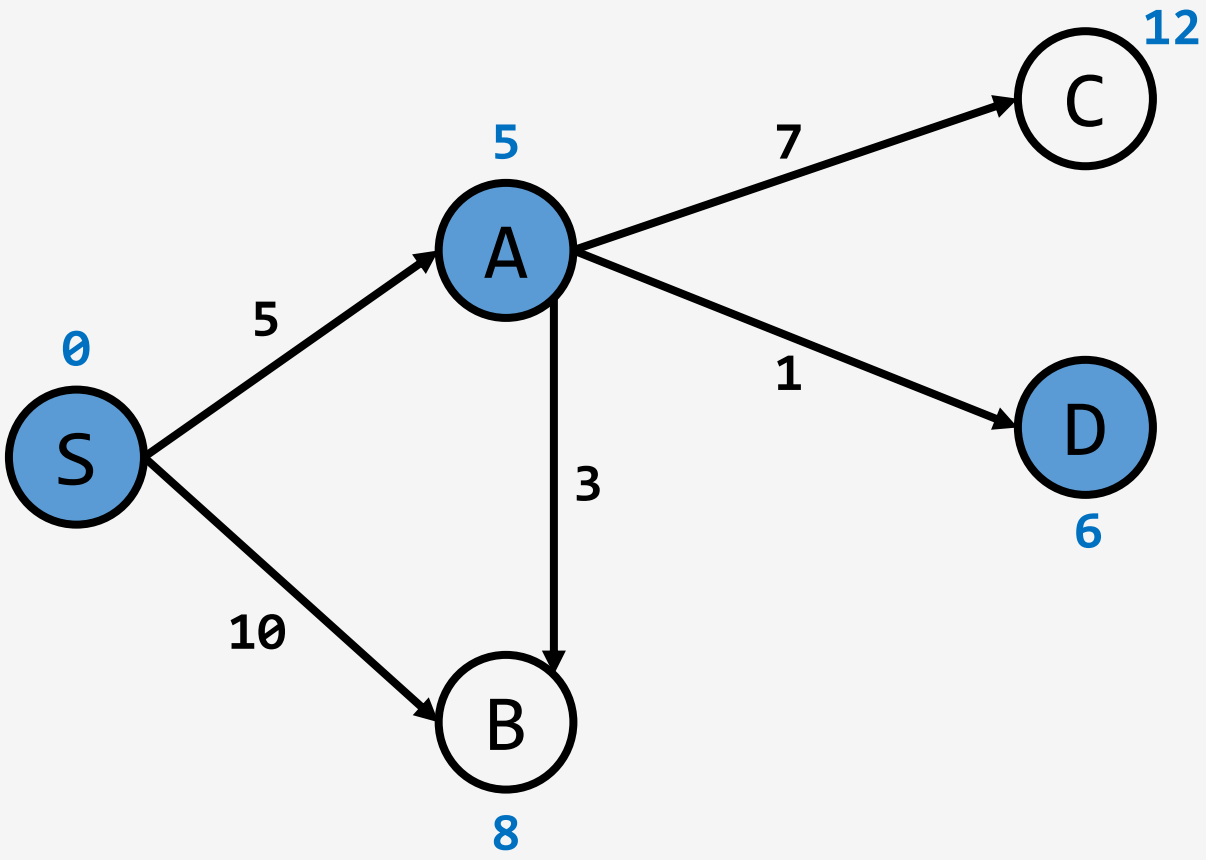
we also discover a few more outgoing edges



what is the next vertex for which we already know the correct distance?

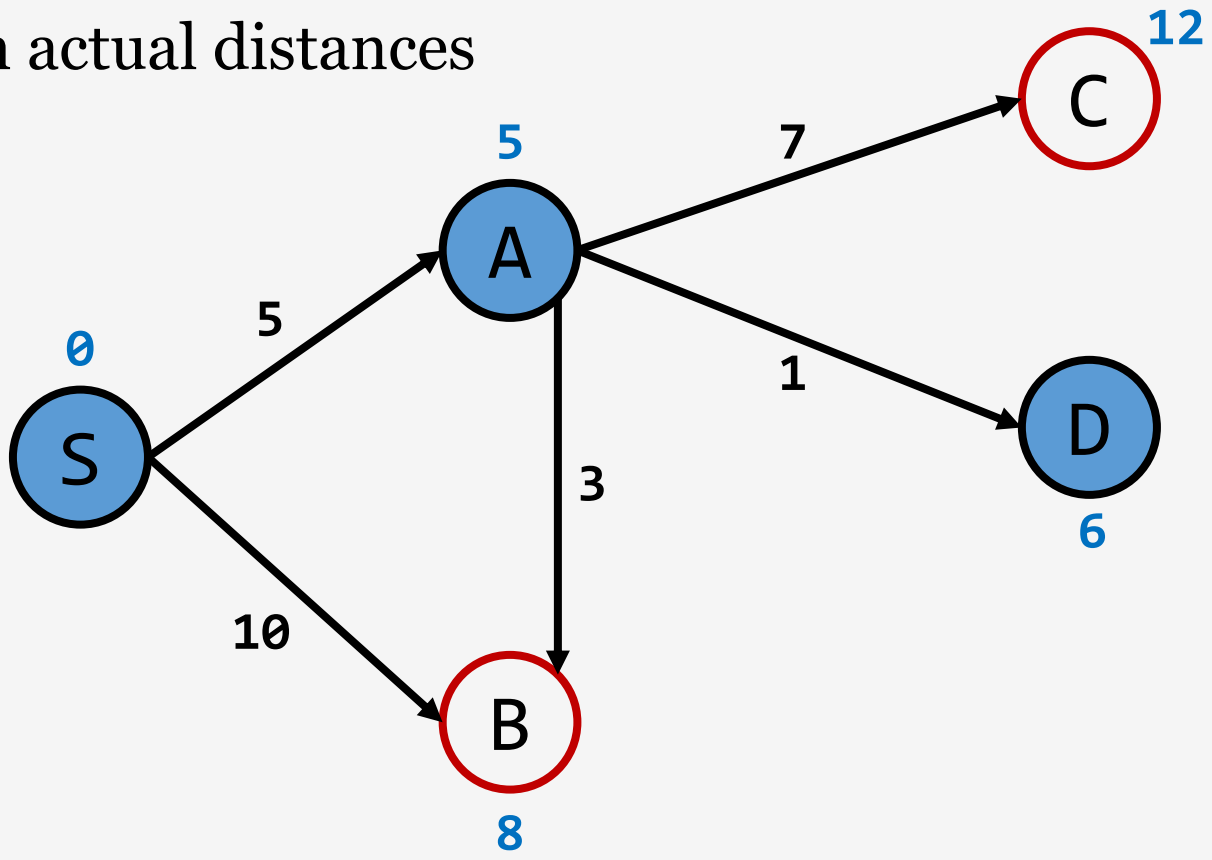
Intuition

It is D



Intuition

while for B and C it is possible that their dist values are larger than actual distances



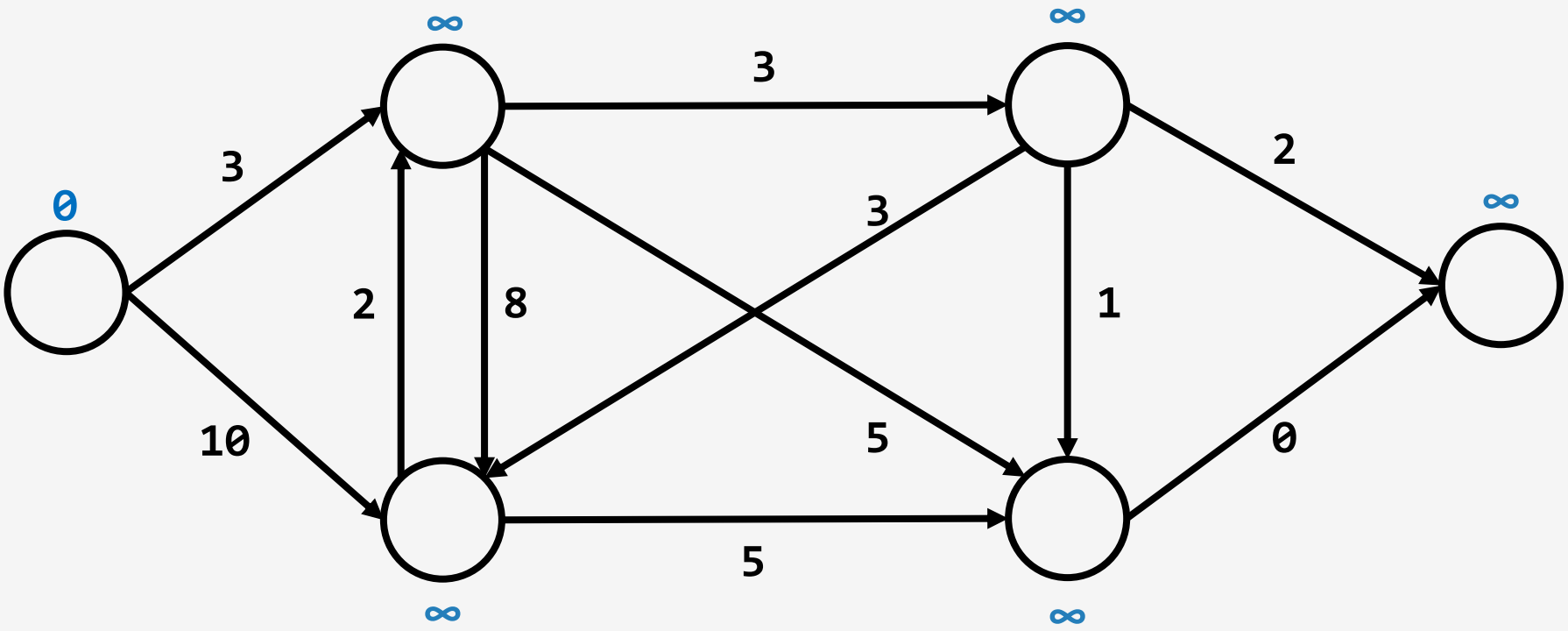
Main ideas of Dijkstra's Algorithm

We maintain a set R of vertices for which dist is already set Correctly ("known region").

The first vertex added to R is S.

On each iteration we take a vertex outside of R with the minimal dist-value, add it to R, and relax all its outgoing edges.

Example: Dijkstra's Algorithm



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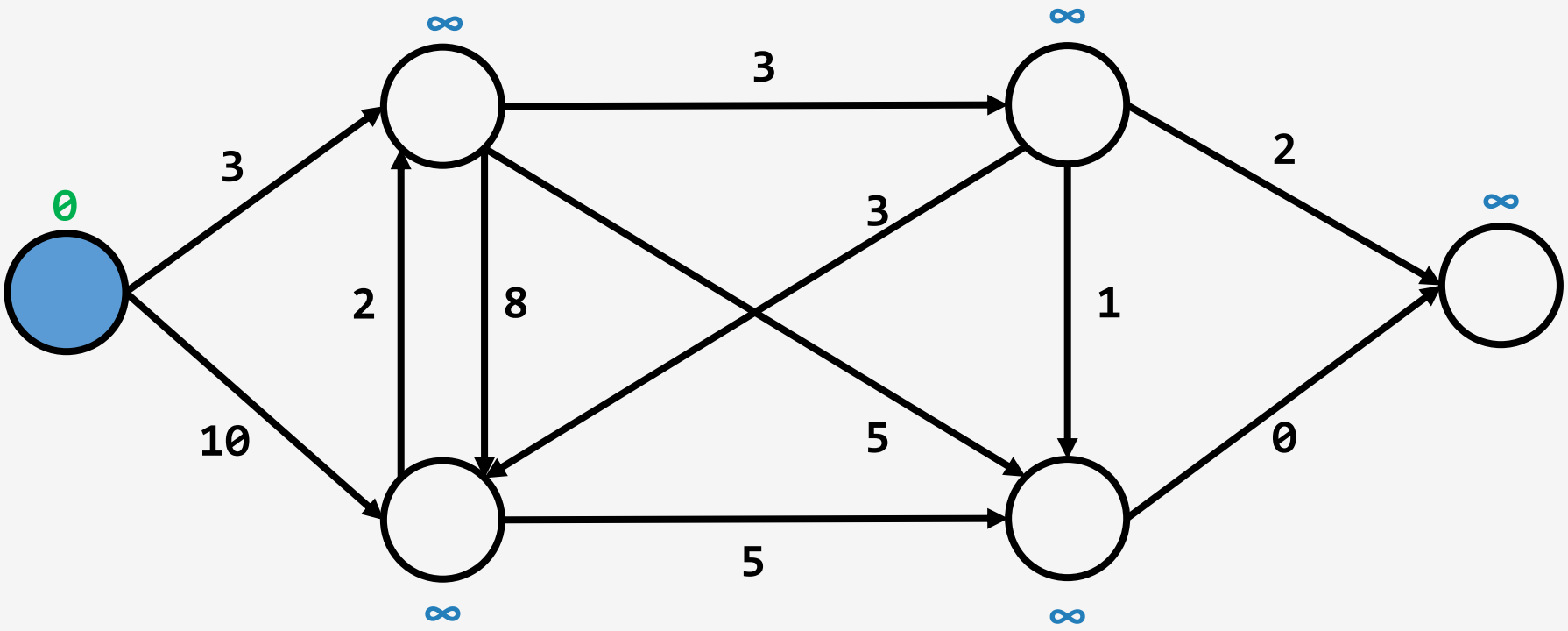

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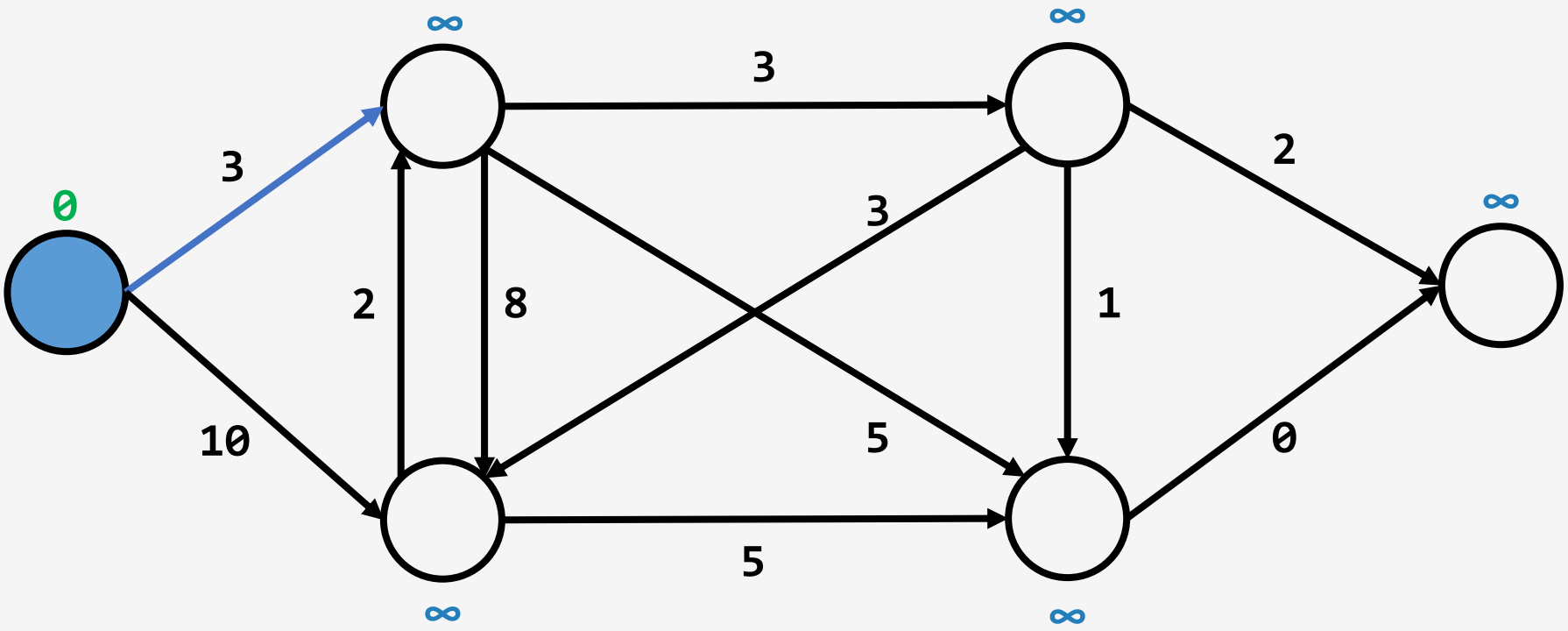

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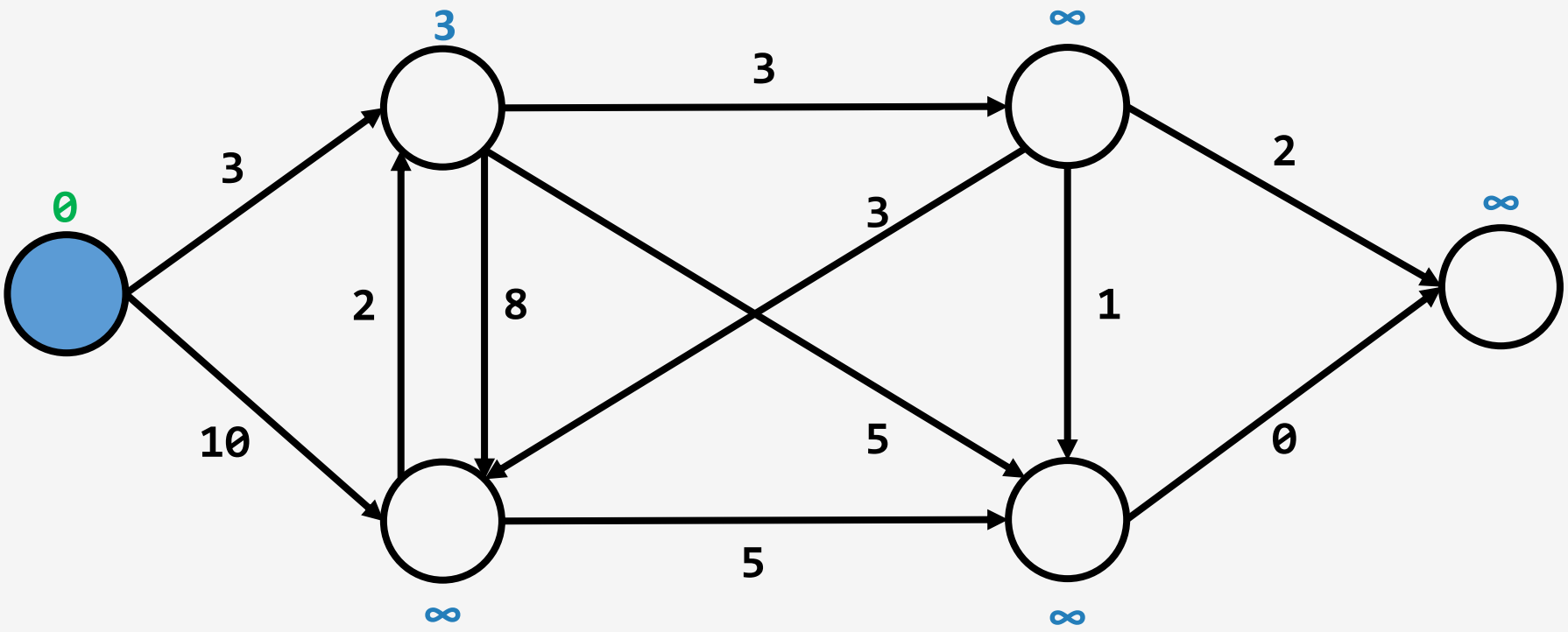

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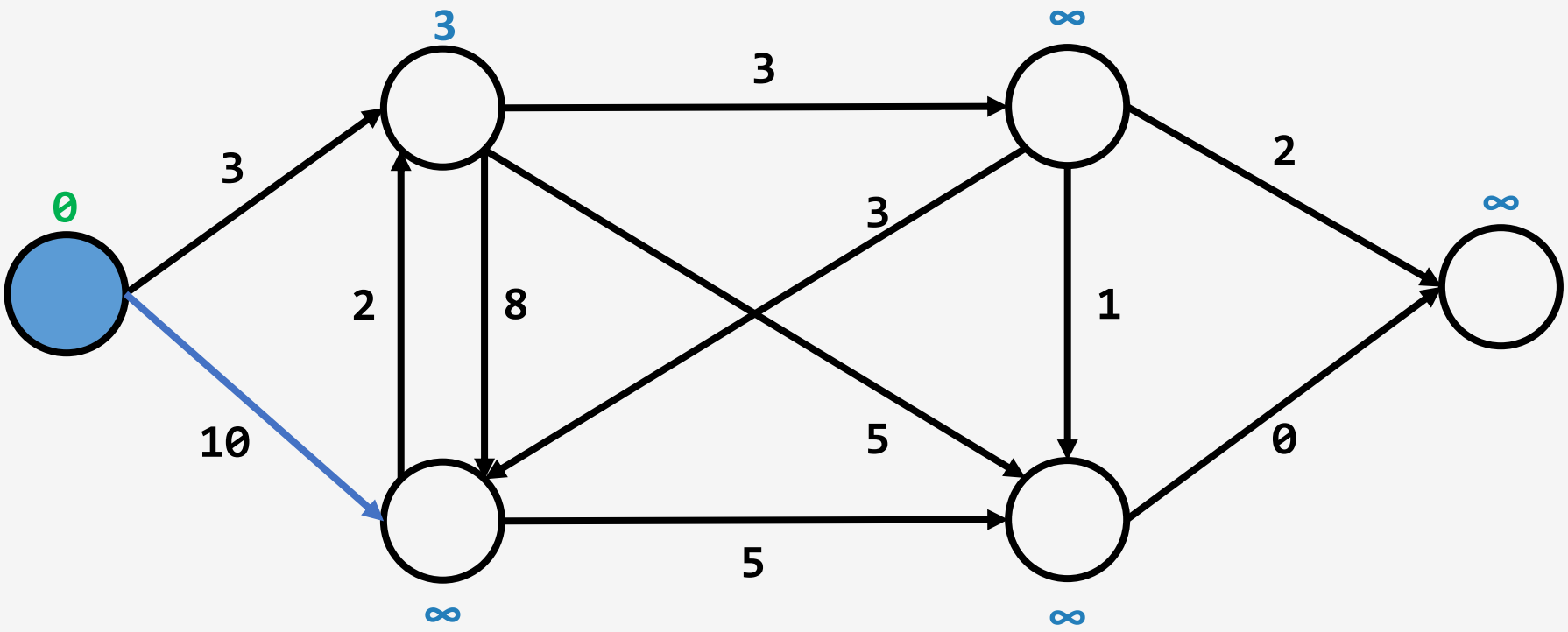

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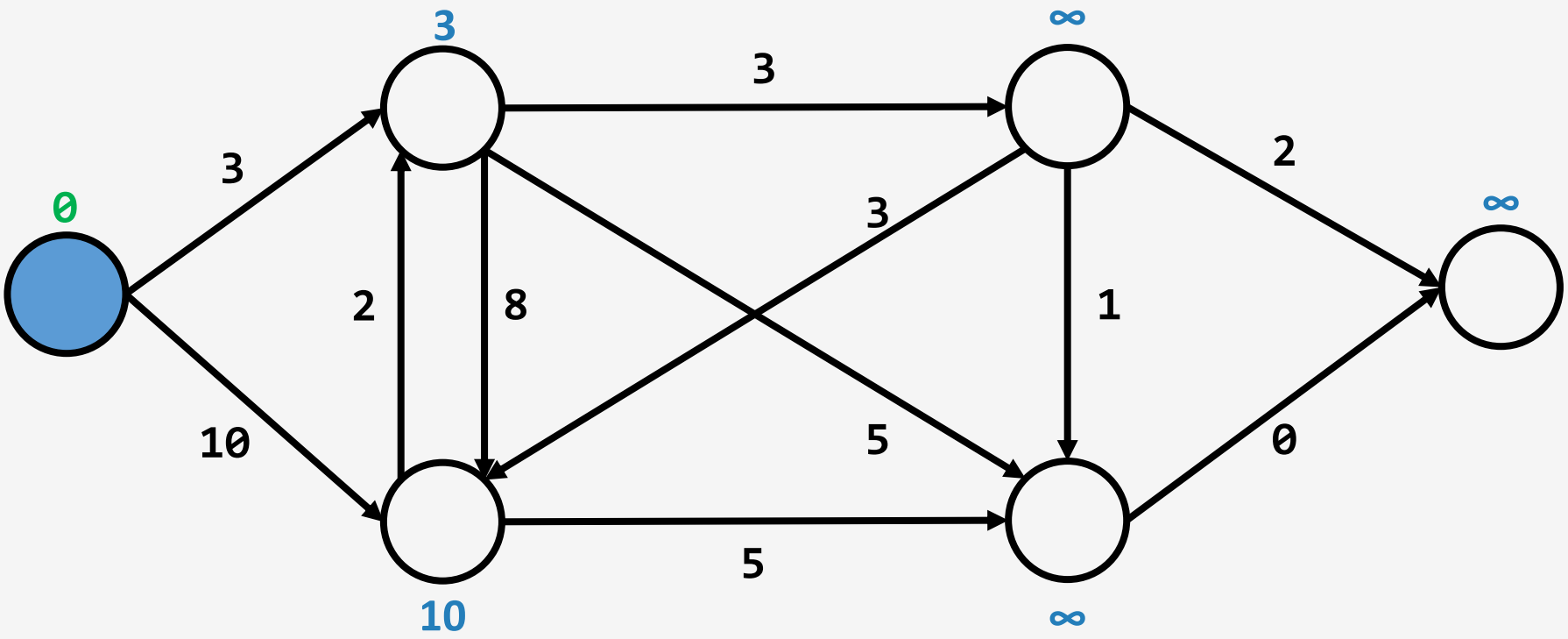

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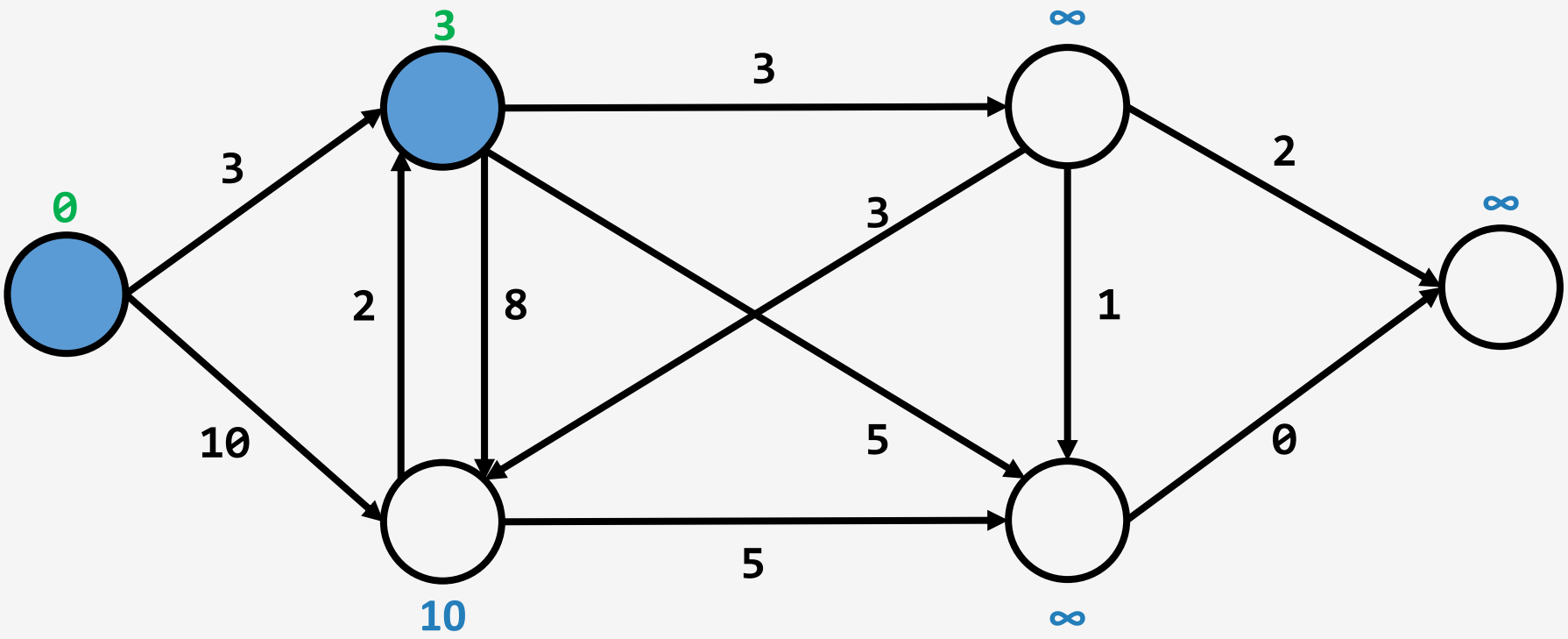

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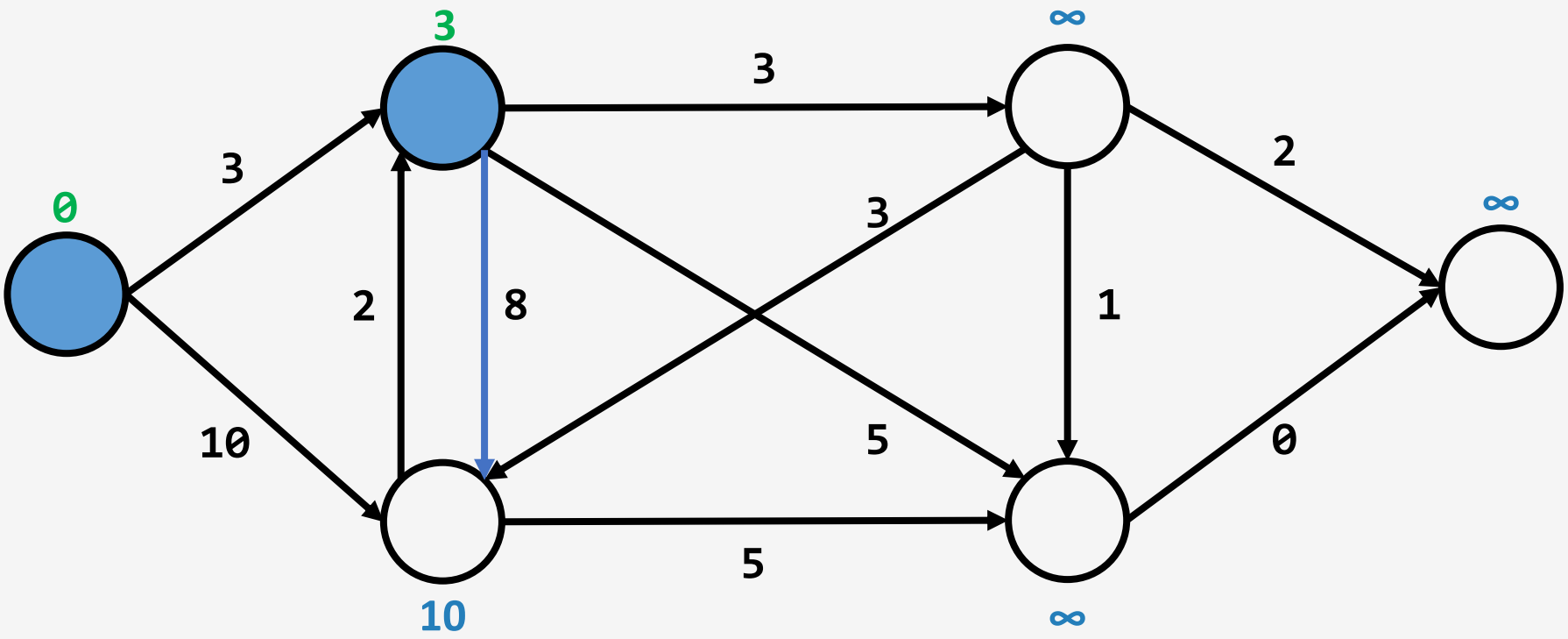

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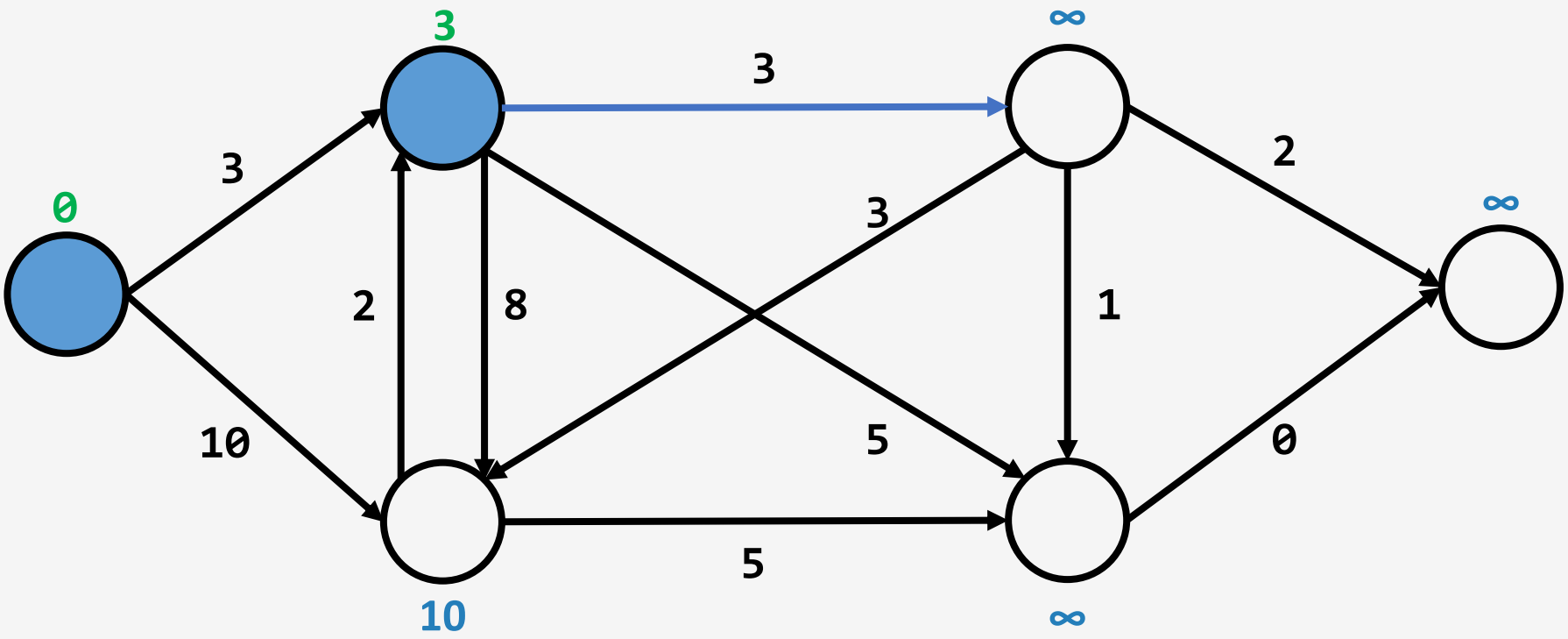

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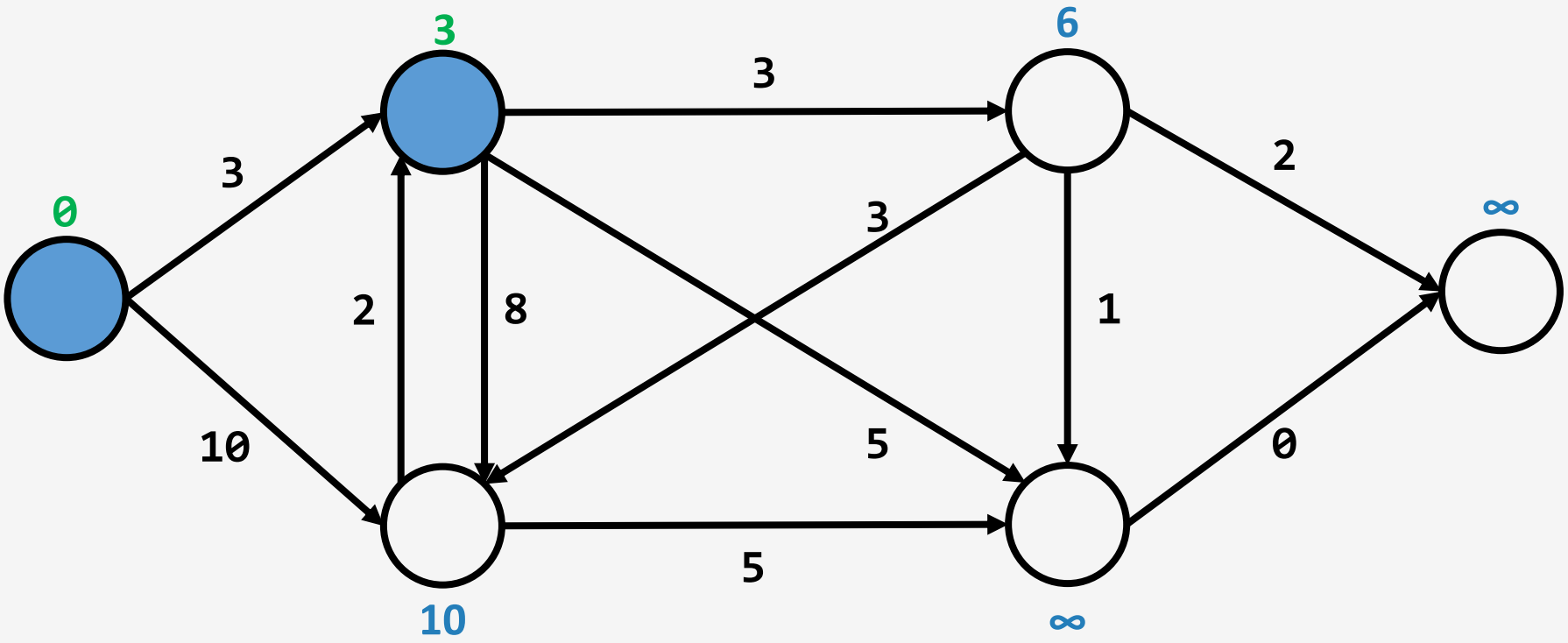

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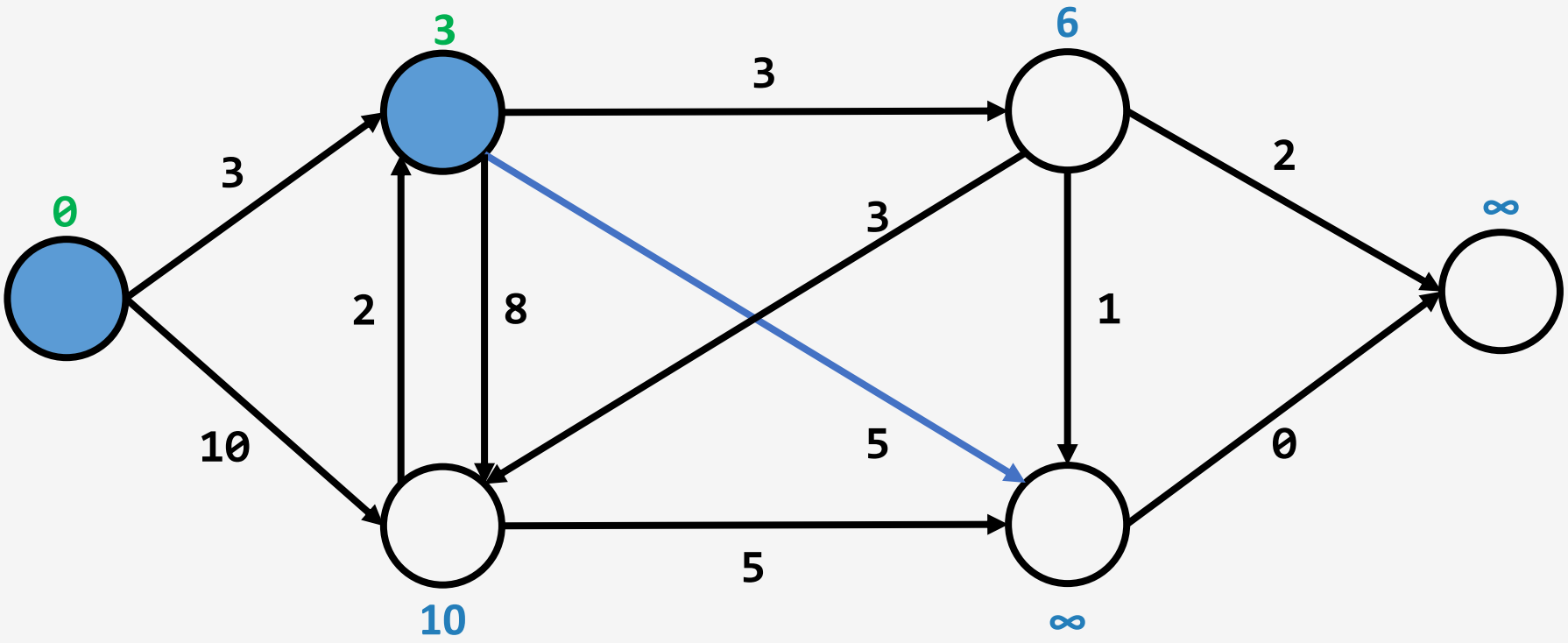

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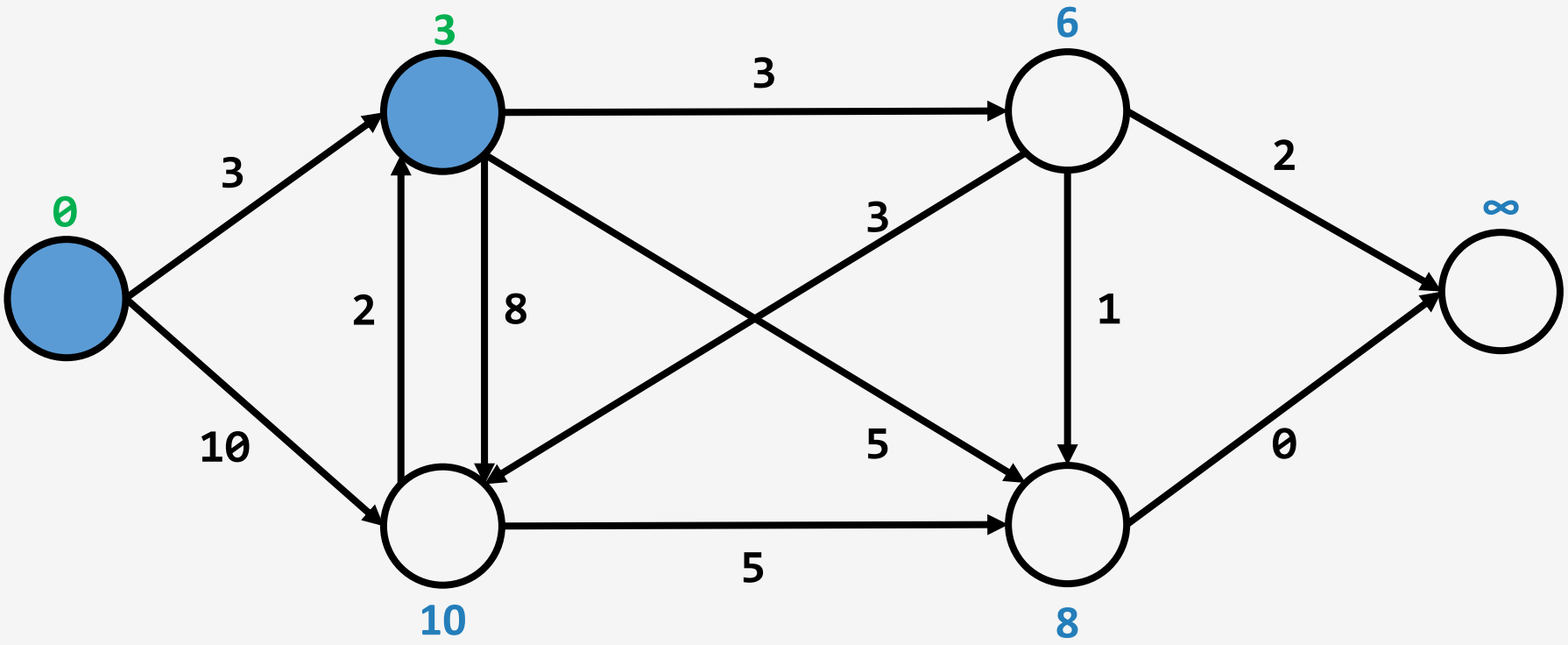

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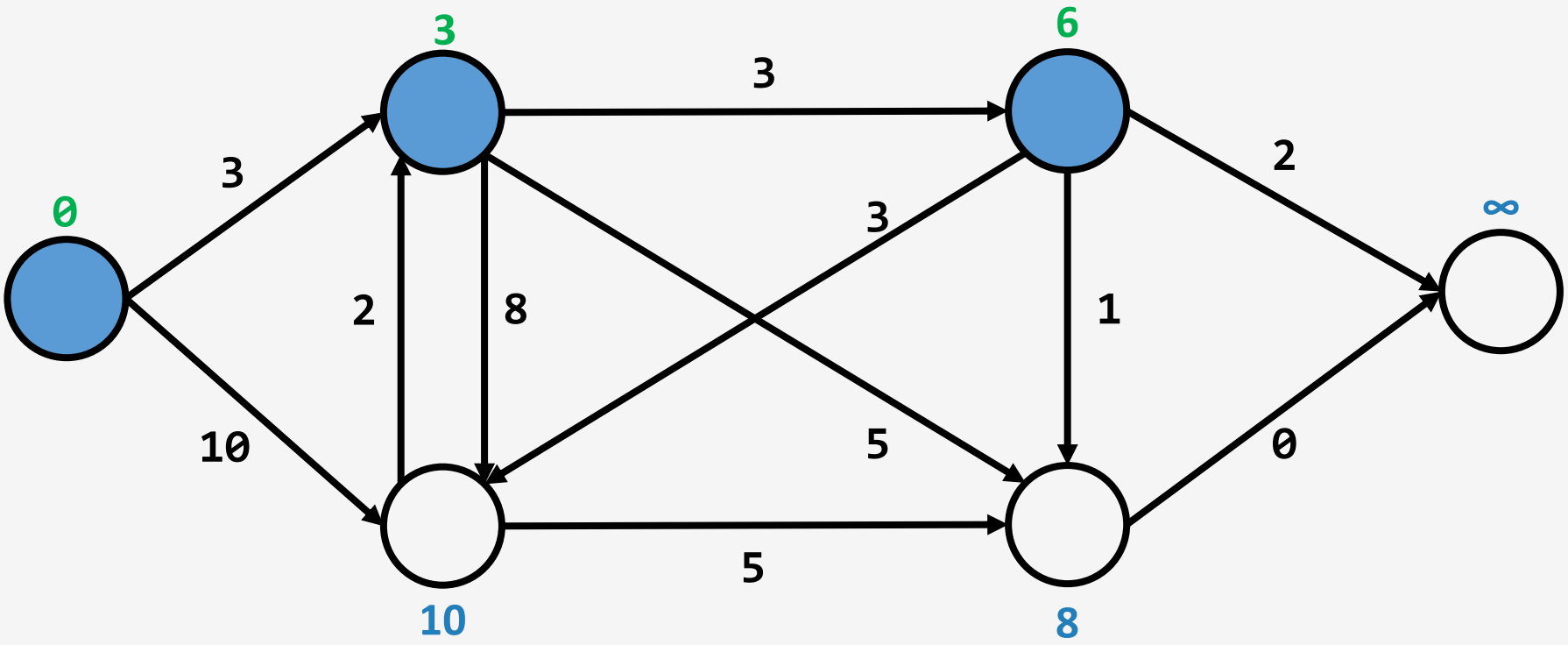

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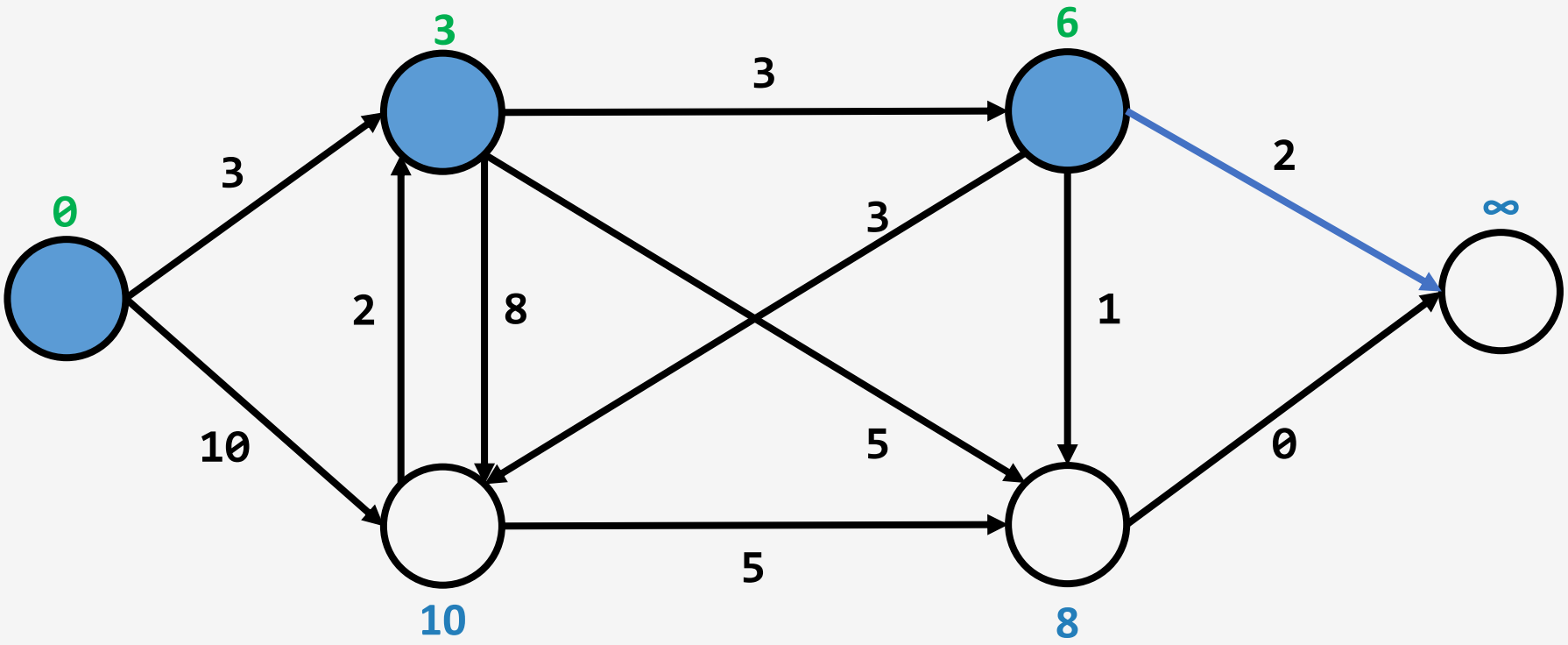

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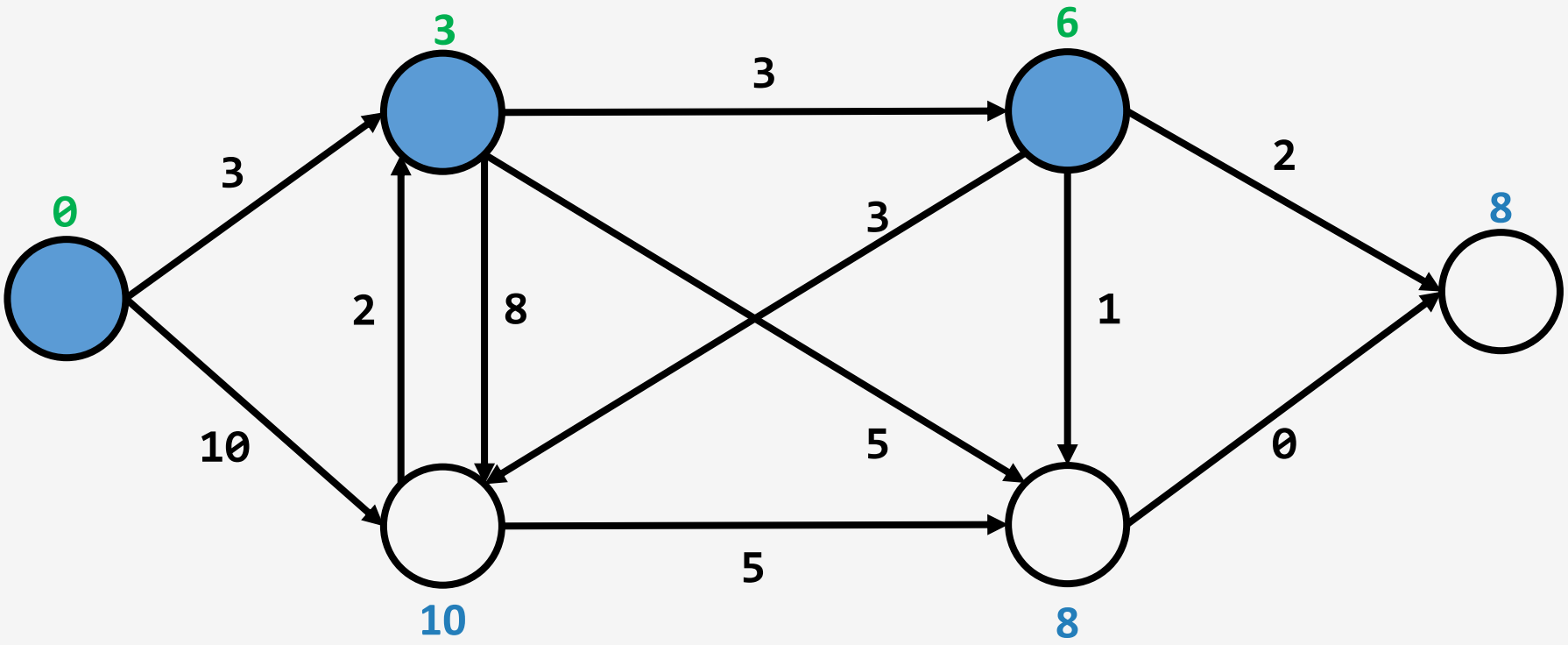

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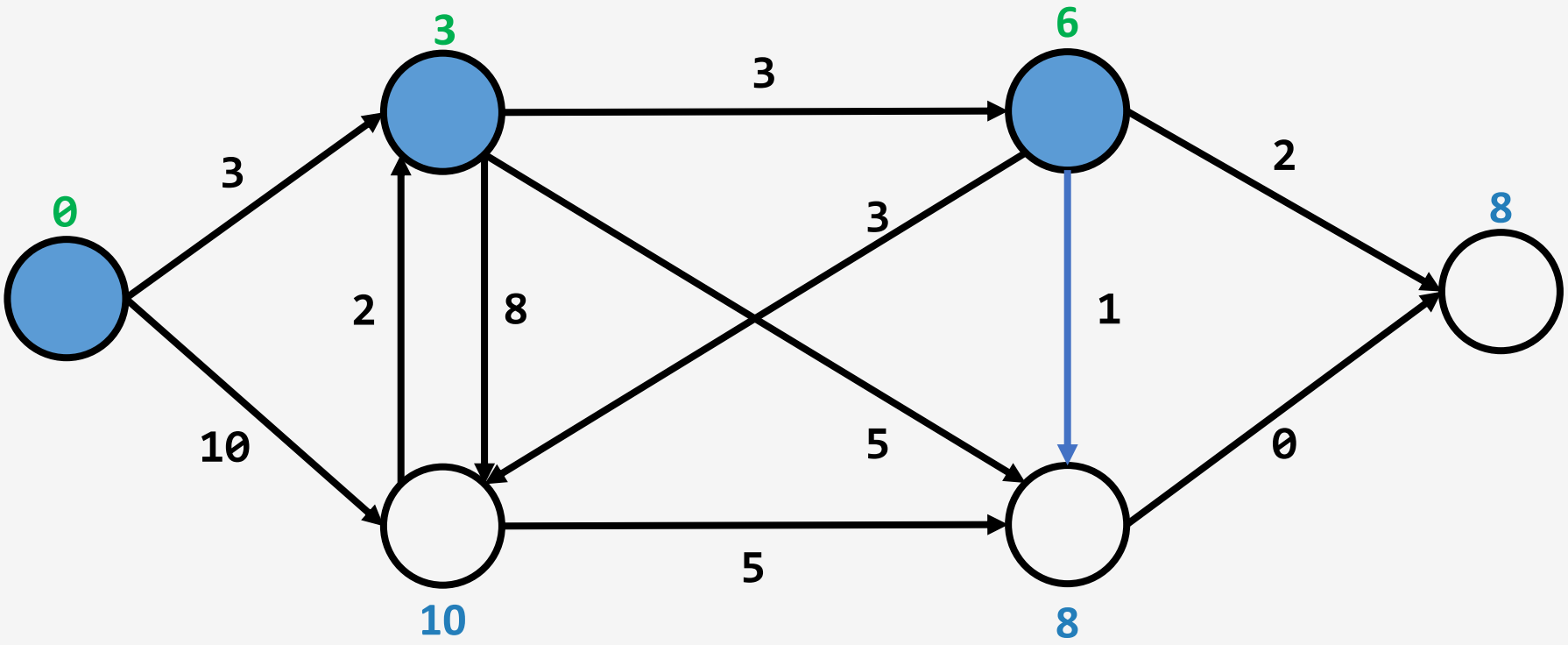

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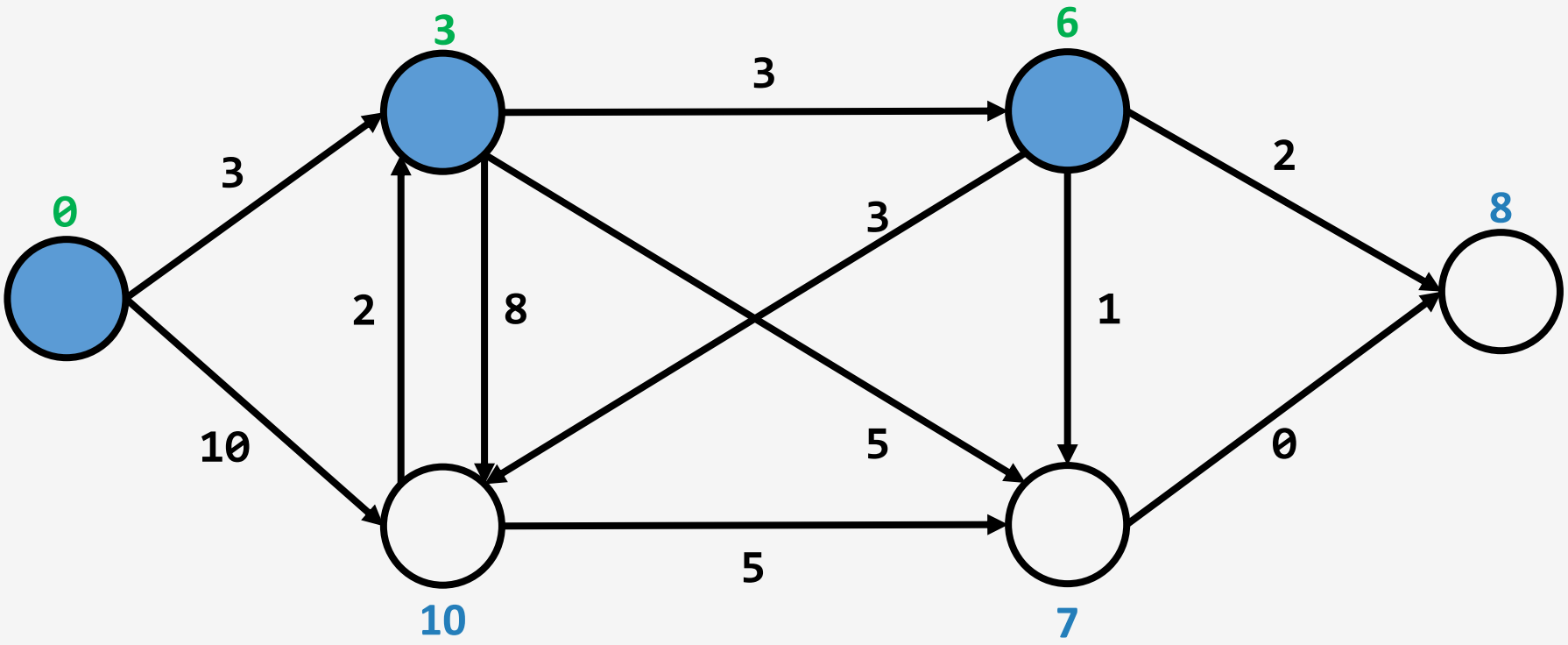

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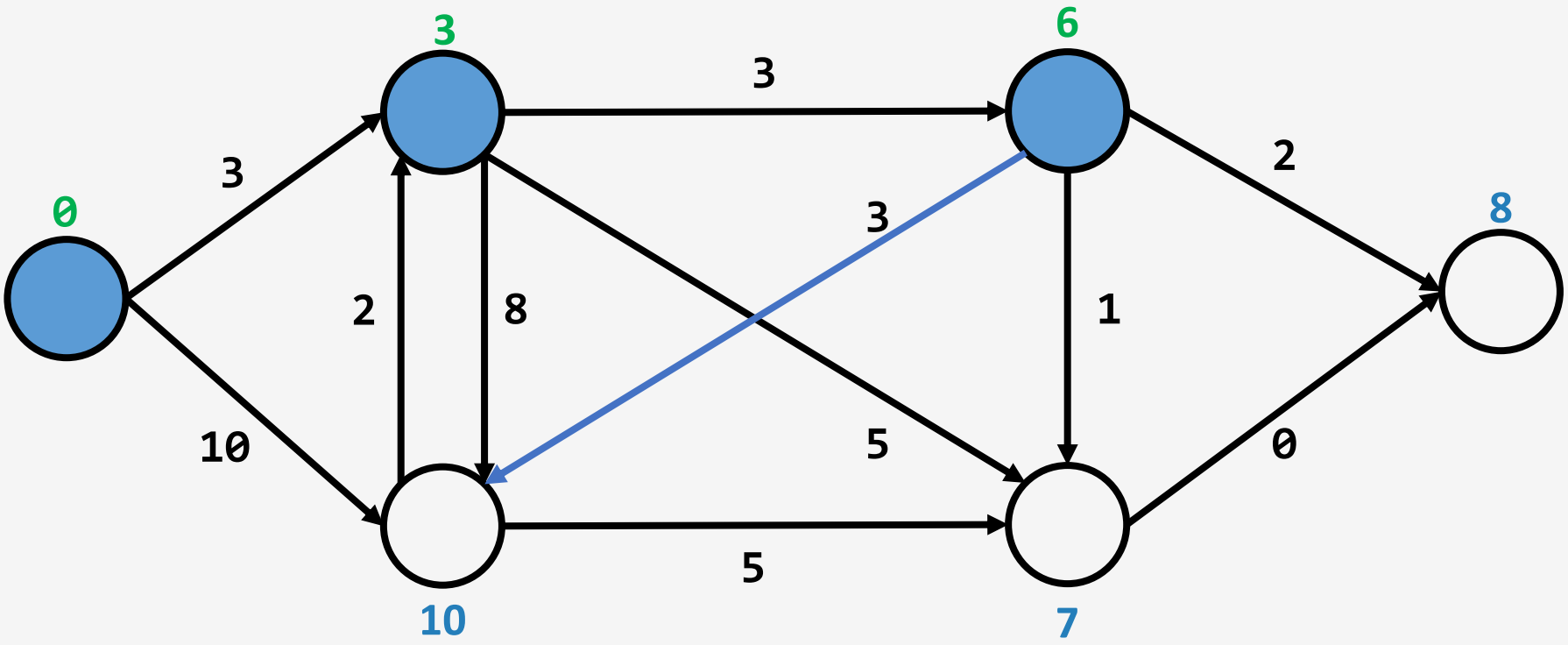

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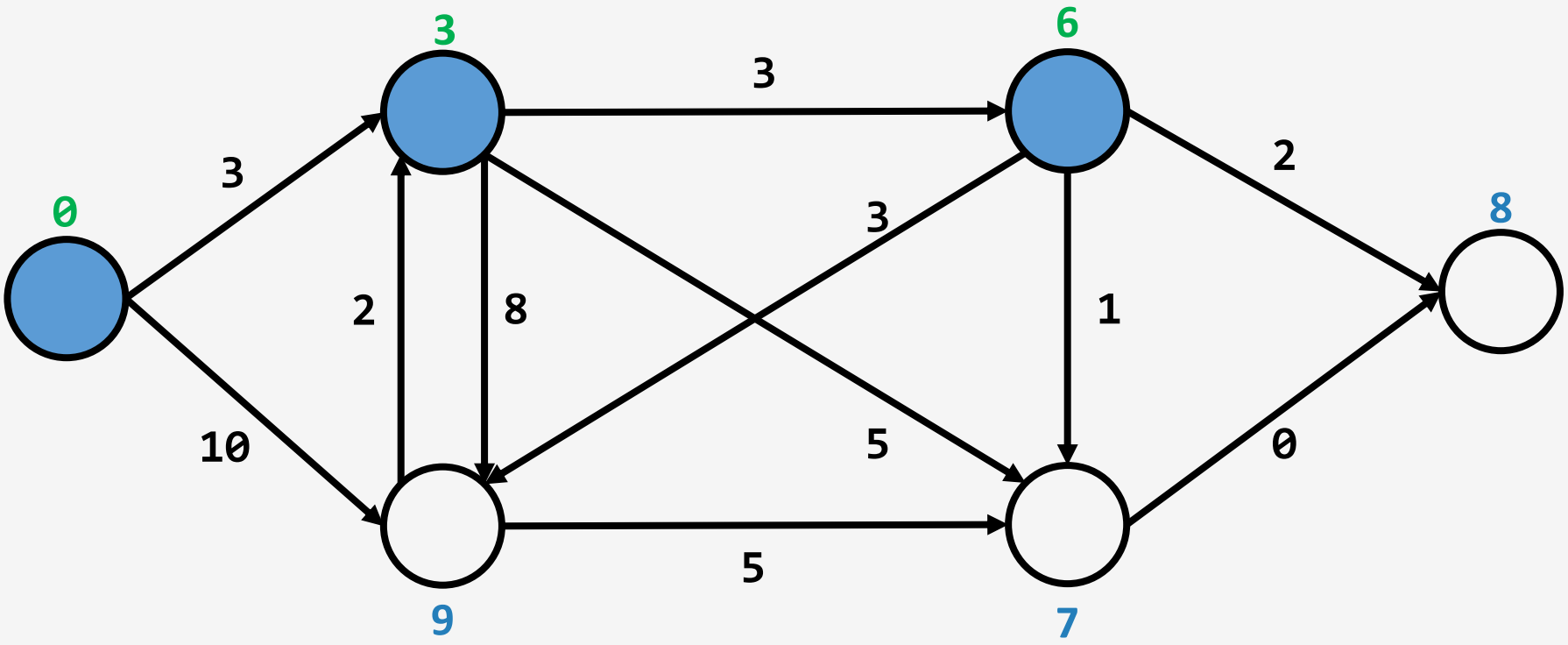

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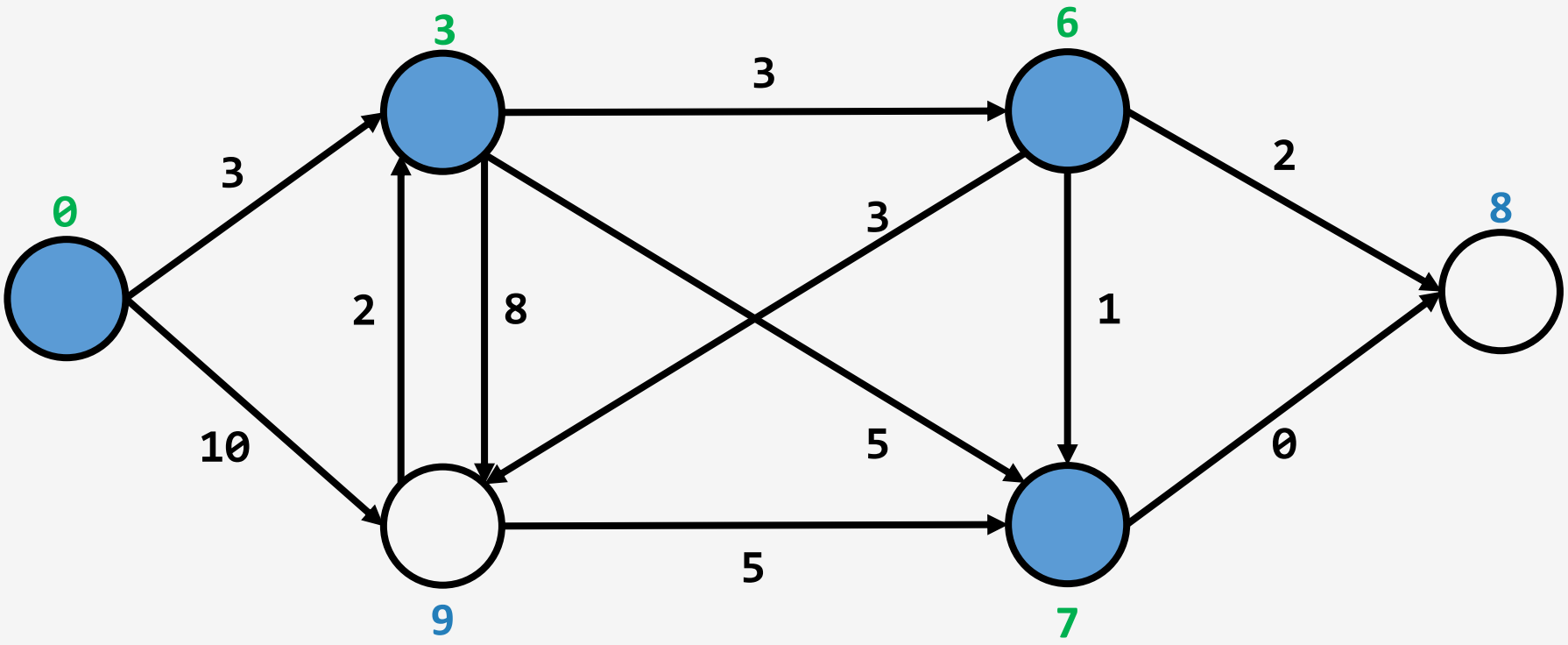

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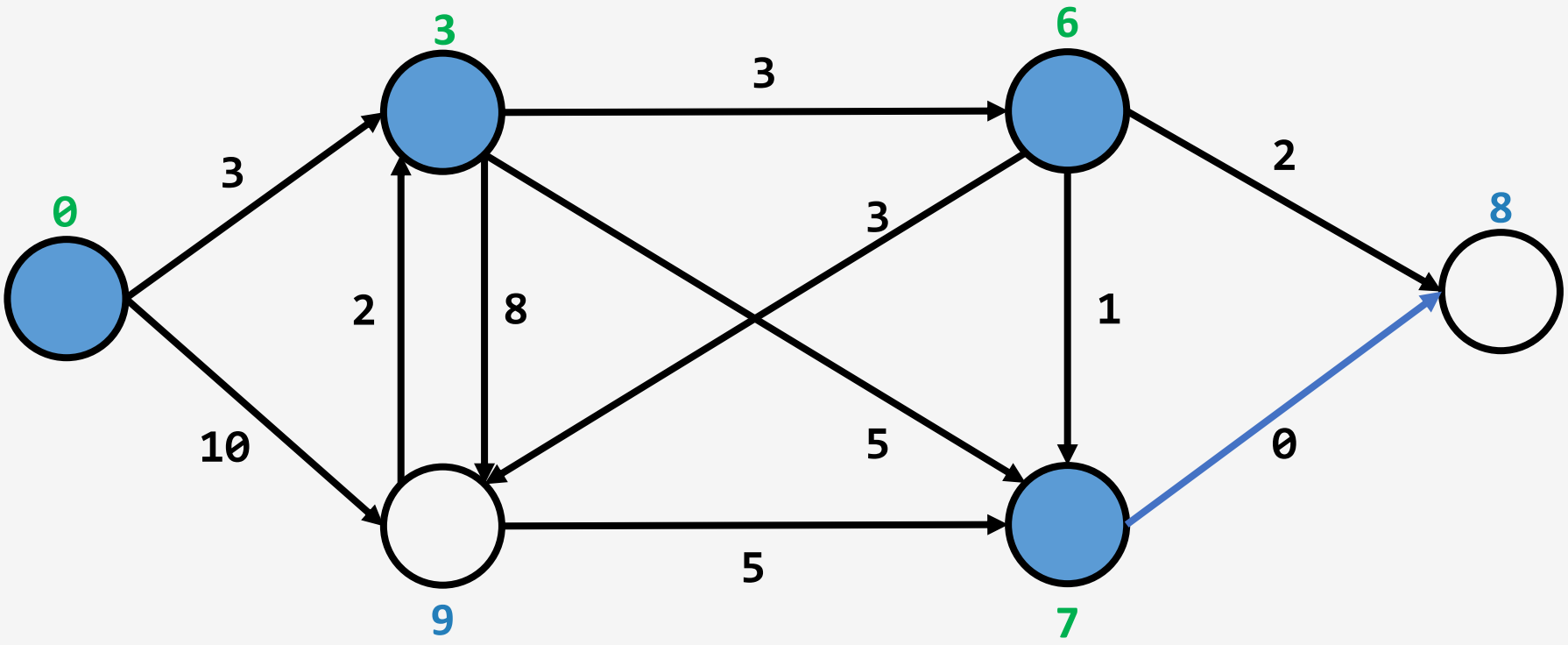

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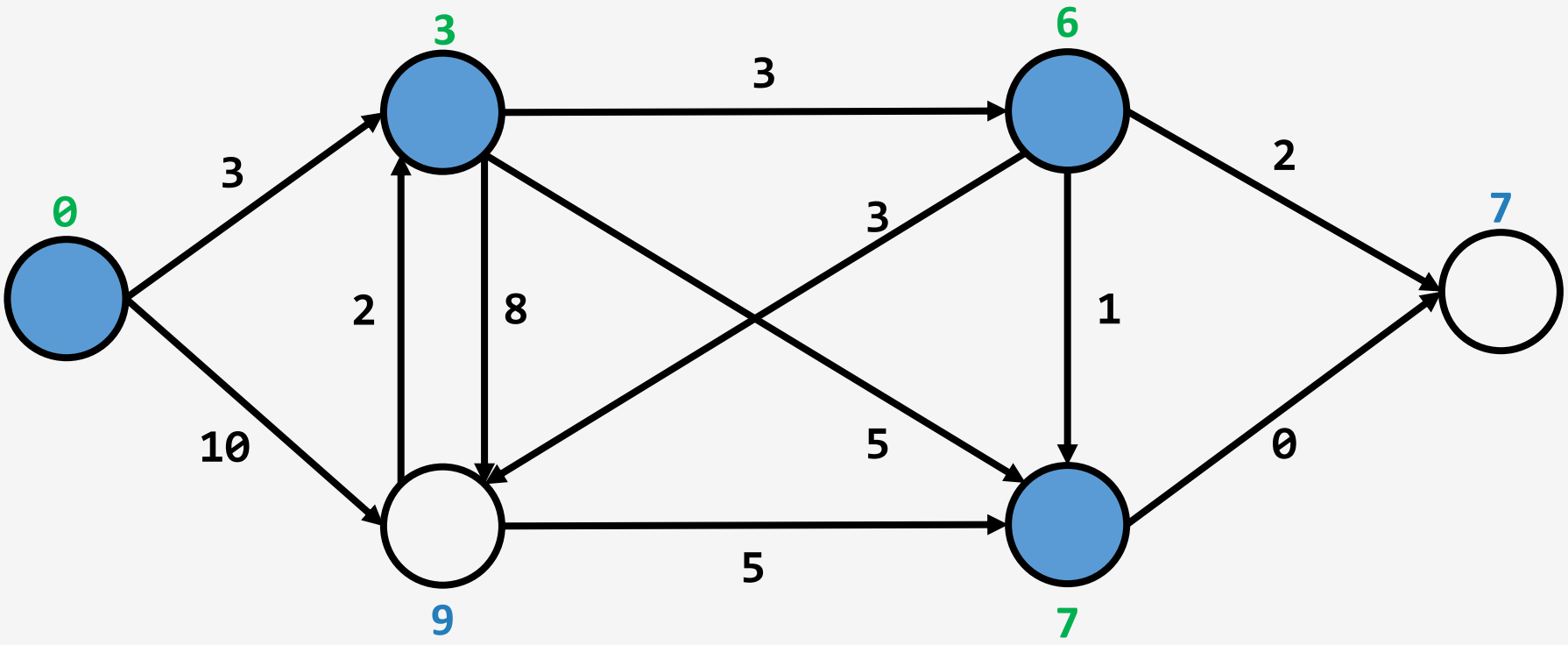

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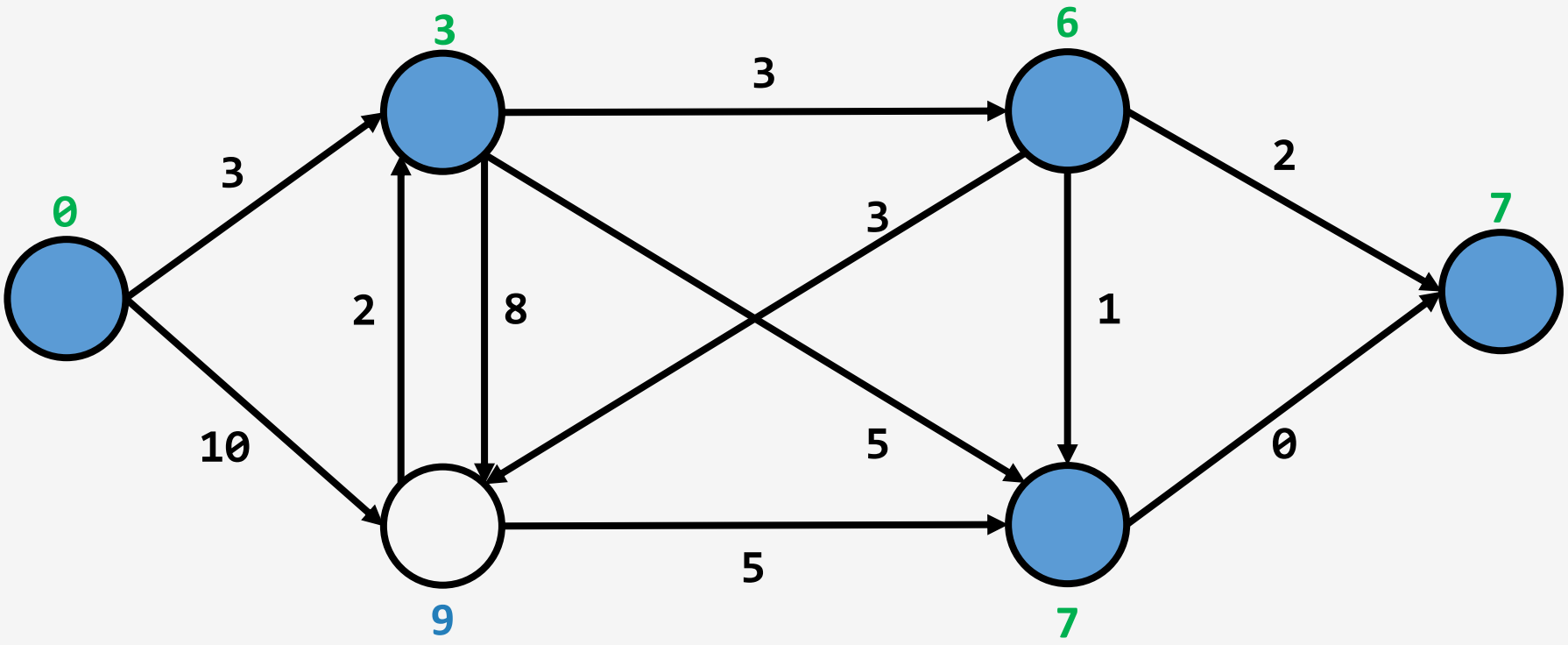

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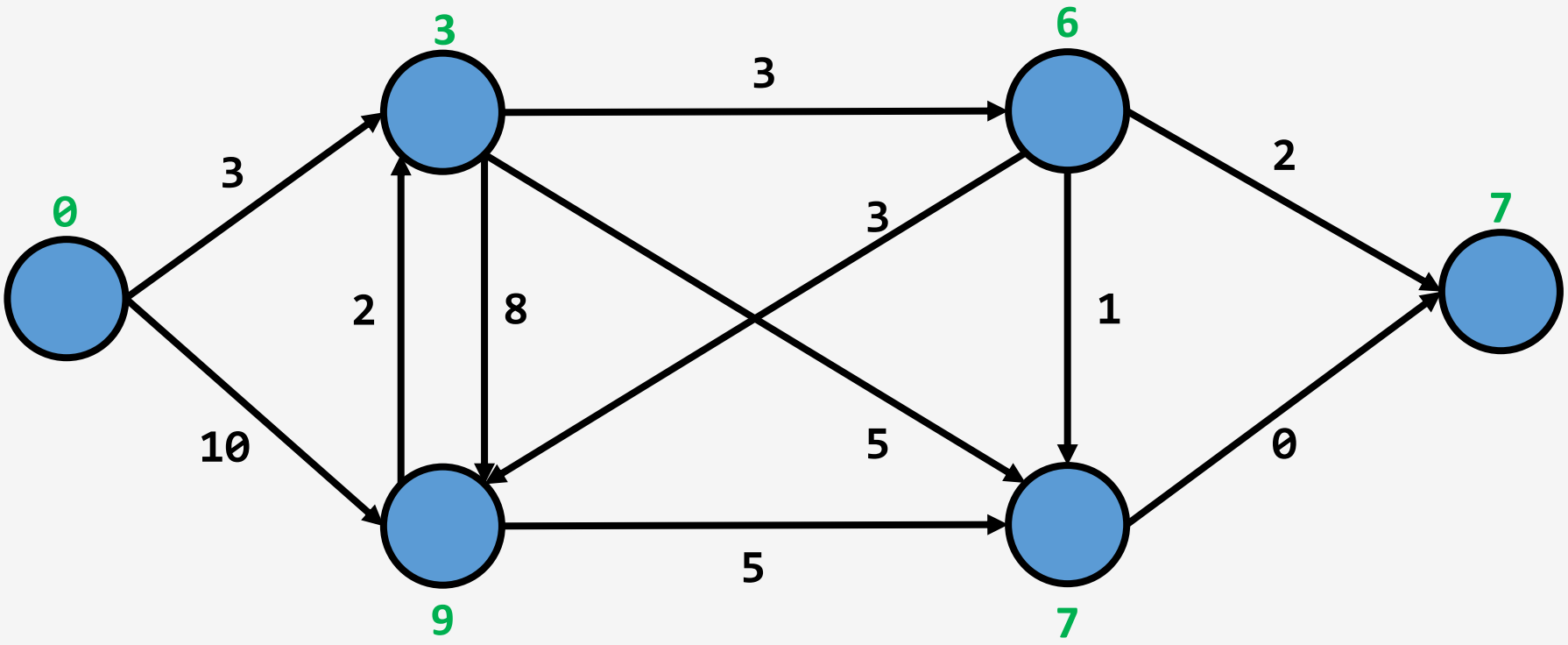

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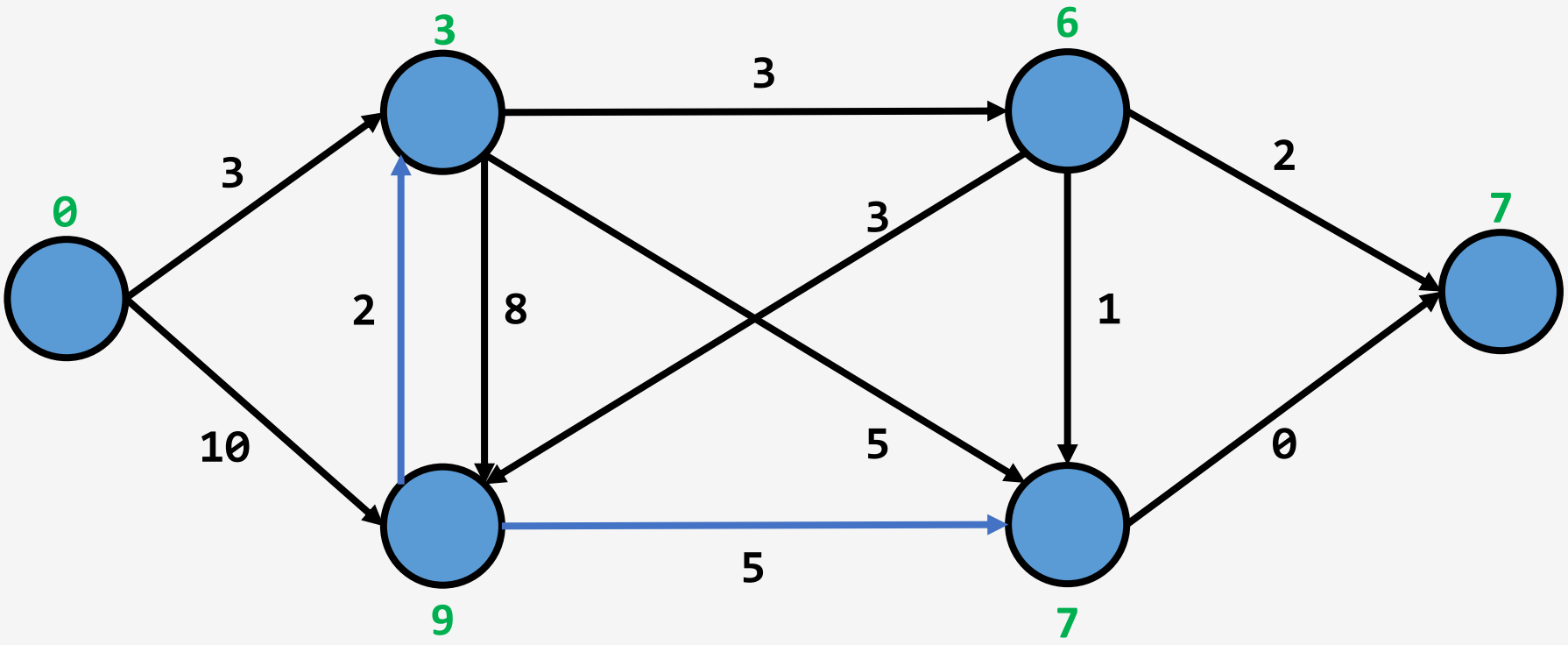

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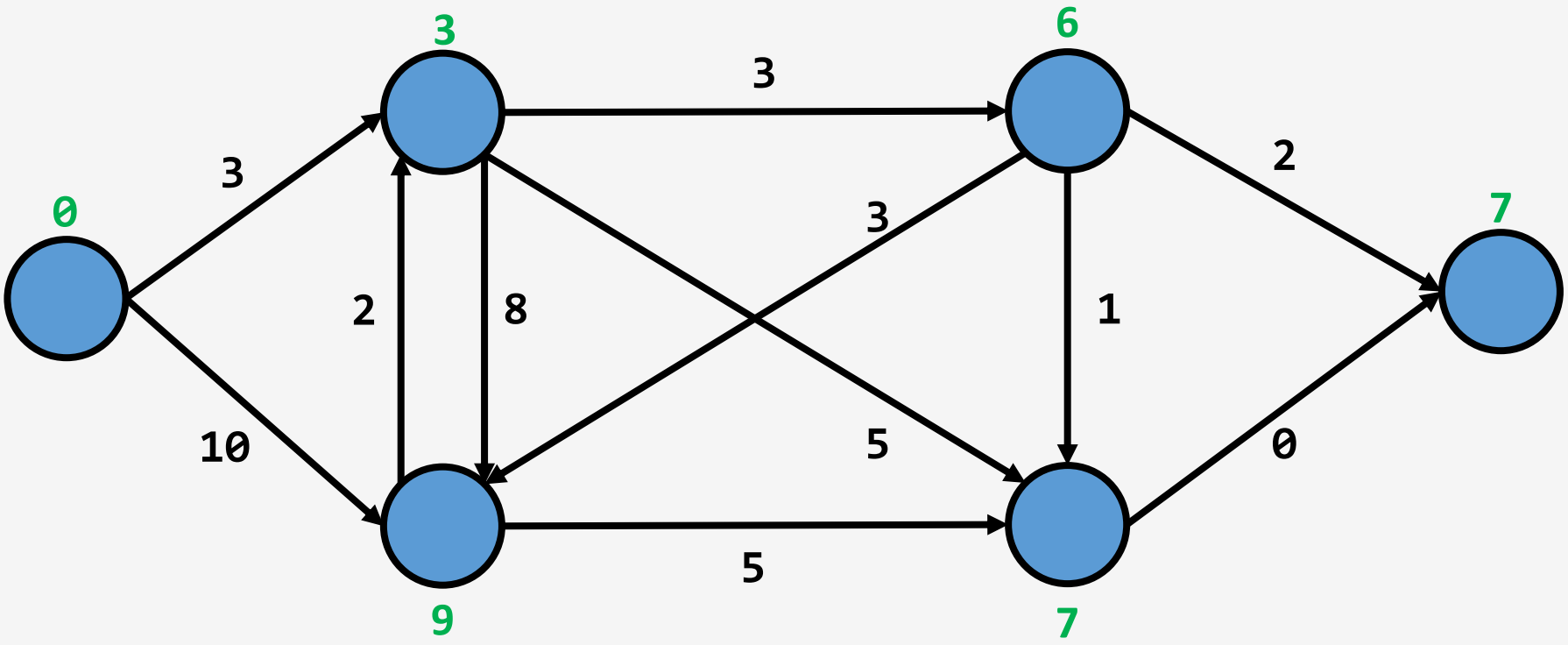

Pseudocode

```
Dijkstra(G, S)
      for all u ∈ V:
         dist[u] ← ∞, prev[u] ← nil
      dist[S] ← 0
      H ← MakeQueue(V) {dist-values as keys}
      while H is not empty:
         u ← ExtractMin(H)
         for all (u, v) ∈ E:
            if dist[v] > dist[u] + w(u, v):
               dist[v] ← dist[u] + w(u, v)
               prev [v] ← u
               ChangePriority(H, v, dist[v]).
```

Conclusion

Can find the minimum time to get from work to home

Can find the fastest route from work to home

Works for any graph with non-negative edge weights

Works in $O(|V|^2)$ or $O((|V| + |E|) \log(|V|))$
depending on the implementation