Design and Analysis of Algorithms

04-04
Dynamic Programming

Placing Parenthesis

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Problem Overview

How to place parentheses in an expression

$$1 + 2 - 3 \times 4 - 5$$

to maximize its value?

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Example:

$$((((1 + 2) - 3) \times 4) - 5) = -5$$

$$((1 + 2) - ((3 \times 4) - 5)) = -4$$

Answer

$$((1 + 2) - (3 \times (4 - 5))) = 6$$

Another example: What about

$$5 - 8 + 7 \times 4 - 8 + 9 \,?$$

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Soon, We'll design an efficient dynamic programming algorithm to find the answer.

Placing Parentheses

Input: A sequence of digits $d_1, \ldots, d_n$ and a sequence of operations $op_1, \ldots, op_{n-1} \in \{+, -, \times\}$

Output: An order of applying these operations that maximizes the value of the expression

$$d_1 \; op_1 \; d_2 \; op_2 \cdots op_{n-1} \; d_n.$$

Intuition

Assume that the last operation in an optimal parenthesizing of

$5 - 8 + 7 \times 4 - 8 + 9$ is $\times$:

$(5 - 8 + 7) \times (4 - 8 + 9)$.

It would help to know optimal values for subexpressions

$5 - 8 + 7$ and $4 - 8 + 9$.

However: We need to keep track for both the minimal
and the maximal values of subexpressions!

Example: $(5 - 8 + 7) \times (4 - 8 + 9)$

$$\min(5 - 8 + 7) = (5 - (8 + 7)) = -10$$

$$\max(5 - 8 + 7) = ((5 - 8) + 7) = 4$$

$$\min(4 - 8 + 9) = (4 - (8 + 9)) = -13$$

$$\max(4 - 8 + 9) = ((4 - 8) + 9) = 5$$

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$$\max((5 - 8 + 7) \times (4 - 8 + 9)) = 130$$

Subproblems

Let $E_{i,j}$ be the subexpression

$$d_i \; op_i \; \cdot \; \cdot \; \cdot \; op_{j-1} \; d_j$$

Subproblems:

$$M(i\,,j) = \text{maximum value of } E_{i,j}$$
$$m(i\,,j) = \text{minimum value of } E_{i,j}$$

Recurrence Relation

$$M(i,j) = \max_{i \le k \le j-1} \begin{cases} M(i,k) \quad op_k \quad M(k+1,j) \\ M(i,k) \quad op_k \quad m(k+1,j) \\ m(i,k) \quad op_k \quad M(k+1,j) \\ m(i,k) \quad op_k \quad m(k+1,j) \end{cases}$$

$$m(i,j) = \min_{i \le k \le j-1} \begin{cases} M(i,k) \quad op_k \quad M(k+1,j) \\ M(i,k) \quad op_k \quad m(k+1,j) \\ m(i,k) \quad op_k \quad M(k+1,j) \\ m(i,k) \quad op_k \quad m(k+1,j) \end{cases}$$

Algorithm

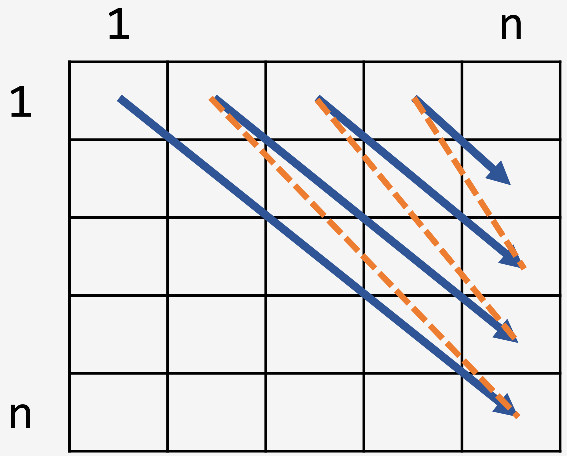
```
MinAndMax(i , j)
      min ← +∞
      max ← −∞
      for k from i to j − 1:
         a ← M(i , k) op_k M(k + 1, j)
         b ← M(i , k) op_k m(k + 1, j)
         c ← m(i , k) op_k M(k + 1, j)
         d ← m(i , k) op_k m(k + 1, j)
         min ← min(min, a, b, c, d)
         max ← max(max, a, b, c, d)
return (min, max)
```

Order of Subproblems

When computing M(i , j), the values of M(i , k) and M(k + 1, j) should be already computed.

Solve all subproblems in order of increasing (j – i).

Possible Order



Algorithm

```
Parentheses(d₁ op₁ d₂ op₂ . . . dₙ)
      for i from 1 to n:
         m(i , i) ← dᵢ, M(i , i) ← dᵢ
      for s from 1 to n − 1:
         for i from 1 to n − s:
            j ← i + s
            m(i , j),M(i , j) ← MinAndMax(i , j)

return M(1, n)
```

Example: $5 - 8 + 7 \times 4 - 8 + 9$

5	-3	-10	-55	-63	-94
	8	15	36	-60	-195
		7	28	-28	-91
			4	-4	-13
				8	17
					9

m

5	-3	4	25	65	200
	8	15	60	52	75
		7	28	20	35
			4	-4	5
				8	17
					9

M

Reconstructing a Solution