Design and Analysis of Algorithms

03-05
Divide – and – Conquer

Master Theorem

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What is the Master Theorem?

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

$$\Downarrow$$

$$T(n) = O(\log n)$$

What is the Master Theorem?

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$\Downarrow$$

$$T(n) = O(n^2)$$

What is the Master Theorem?

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$\Downarrow$$

$$T(n) = O\left(n^{\log_2 3}\right)$$

What is the Master Theorem?

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$\Downarrow$$

$$T(n) = O(n \log n)$$

Master Theorem

$$if\ T(n) = aT\left(\left\lceil\frac{n}{2}\right\rceil\right) + O(n^d)$$

(for constants: $a > 0, b > 1, d \geq 0) then$

$$T(n) = \begin{cases} O(n^d) & if\ d > \log_b a \\ O(n^d \log n) & if\ d = \log_b a \\ O(n^{\log_b a}) & if\ d < \log_b a \end{cases}$$

Master Theorem: Example 1

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

$$a = 4, b = 2, d = 1$$

$$Since \quad d < \log_a b$$

$$T(n) = O\left(n^{\log_b a}\right) = O(n^2)$$

Master Theorem: Example 2

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

$$a = 3, b = 2, d = 1$$

$$Since \quad d < \log_a b$$

$$T(n) = O\left(n^{\log_b a}\right) = O\left(n^{\log_2 3}\right)$$

Master Theorem: Example 3

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$a = 2, b = 2, d = 1$$

$$Since \quad d = \log_a b$$

$$T(n) = O\left(n^d \log n\right) = O(n \log n)$$

Master Theorem: Example 4

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

$$a = 1, b = 2, d = 0$$

$$Since \quad d = \log_a b$$

$$T(n) = O\left(n^d \log n\right) = O(n^0 \log n) = O(\log n)$$

Master Theorem: Example 5

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n^2)$$

$$a = 2, b = 2, d = 2$$

$$Since \quad d > \log_a b$$

$$T(n) = O\left(n^d\right) = O(n^2)$$

$$T(n) = aT\left(\left\lceil \frac{n}{2} \right\rceil\right) + O\left(n^d\right)$$

Level		#	Work
0	n	1	$O\left(n^d\right)$
1	n/b $\cdots$ n/b	a	$aO\left(\frac{n}{b}\right)^d = O\left(n^d\right)\frac{a}{b^d}$
$\vdots$	$\vdots$		$\vdots \quad \vdots$
i	$\cdots$ n/b^i $\cdots$	a^i	$a^i O\left(\frac{n}{b^i}\right)^d = O\left(n^d\right)\left(\frac{a}{b^d}\right)^i$
$\vdots$	$\vdots$		$\vdots \quad \vdots$
$\log_b n$	1 $\cdots$ 1	$a^{\log_b n}$	$a^{\log_b n} = O\left(n^{\log_b a}\right)$

$$Total = \sum_{i=0}^{\log_b n} O\left(n^d\right)\left(\frac{a}{b^d}\right)^i$$

Geometric Series

$$For\ \ r \neq 1$$

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a\frac{1 - r^n}{1 - r}$$

$$T(n) = \begin{cases} O(a) & if\ r < 1 \\ O(an^{n-1}) & if\ r > 1 \end{cases}$$

Case 1: $\frac{a}{b^d} < 1 \approx (d > \log_b a)$

$$\sum_{i=0}^{\log_b n} O(n^d)\left(\frac{a}{b^d}\right)^i$$

$$= O(n^d)$$

Case 2: $\frac{a}{b^d} = 1 \approx (d = \log_b a)$

$$\sum_{i=0}^{\log_b n} O(n^d)\left(\frac{a}{b^d}\right)^i$$

$$= \sum_{i=0}^{\log_b n} O(n^d)$$

$$= (1 + \log_b n)\, O(n^d)$$

$$= O(n^d \log n)$$

Case 2: $\frac{a}{b^d} > 1 \approx (d < \log_b a)$

$$\sum_{i=0}^{\log_b n} O(n^d)\left(\frac{a}{b^d}\right)^i$$

$$= O\left(O(n^d)\left(\frac{a}{b^d}\right)^{\log_b n}\right)$$

$$= O\left(O(n^d)\frac{a^{\log_b n}}{b^{d\log_b n}}\right)$$

$$= O\left(O(n^d)\frac{n^{\log_b a}}{n^d}\right)$$

$$= O\left(n^{\log_b a}\right)$$

Summary

Master theorem is a shortcut