Design and Analysis of Algorithms

03-04
Divide – and – Conquer

Polynomial Multiplication

Imran Ihsan
Assistant Professor, Department of Computer Science
Air University, Islamabad, Pakistan
www.imranihsan.com

Uses of Multiplying Polynomials

Error-correcting codes

Large-integer multiplication

Generating functions

Convolution in signal processing

Multiplying Polynomials

Example

$$A(x) = 3x^2 + 2x + 5$$

$$B(x) = 5x^2 + x + 2$$

$$A(x)B(x) = 15x^4 + 13x^3 + 33x^2 + 9x \quad + 10$$

Multiplying Polynomials

Input: Two n – 1 degree polynomials:

$$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \dots a_1 x + a_0$$

$$b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \dots b_1 x + b_0$$

Output: The product polynomial:

$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + c_{2n-4}x^{2n-4} + \dots c_1 x + c_0$$

where

$$c_{2n-2} = a_{n-1}b_{n-1}$$

$$c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$$

…

$$c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$$

$$c_1 = a_1 b_0 + a_0 b_1$$

$$c_0 = a_0 b_0$$

Multiplying Polynomials

Example:

Input: n = 3, A = (3, 2, 5), B = (5, 1, 2)

$$A(x) = 3x^2 + 2x + 5$$

$$B(x) = 5x^2 + x + 2$$

$$A(x)B(x) = 15x^4 + 13x^3 + 33x^2 + 9x + 10$$

Output: C = (15, 13, 33, 9, 10)

Naïve Algorithm

```
MultPoly(A, B, n)
      pair ← Array[n][n]
      for i from 0 to n – 1:
         for j from 0 to n – 1:
            pair [i][j] ← A[i] * B[j]
      product ← Array[2n – 1]
      for i from 0 to 2n – 1:
         product[i] ← 0
      for i from 0 to n – 1:
         for j from 0 to n – 1:
            product[i + j] ← product[i + j] + pair [i][j]
      return product
```

Naïve Solution: $O(n^2)$

Multiply all $a_i b_j$ pairs (n^2 multiplications)

Sum needed pairs (n^2 additions)

Multiplying Polynomials

Let $A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$ *where*

$D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + \ldots + a_{\frac{n}{2}}$

$D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \ldots + a_0$

Let $B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$ *where*

$E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + \ldots + b_{\frac{n}{2}}$

$E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \ldots + b_0$

$$AB = \left(D_1x^{\frac{n}{2}} + D_0\right)\left(E_1x^{\frac{n}{2}} + E_0\right)$$

$$(D_1E_1)x^n + (D_1E_0 + D_0E_1)x^{\frac{n}{2}} + (D_0E_0)$$

Calculate D_1E_1, D_1E_0, D_0E_1, and D_0E_0

Recurrence

$$T(n) = 4\mathrm{T}\left(\frac{n}{2}\right) + kn$$

Polynomial Mult: Divide & Conquer

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

Polynomial Mult: Divide & Conquer

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

$D_1(x) = 4x + 3$

Polynomial Mult: Divide & Conquer

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$D_1 E_1 = 4x^2 + 11x + 6$

Polynomial Mult: Divide & Conquer

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$E_0(x) = 3x + 4$

$D_1E_1 = 4x^2 + 11x + 6$

$D_1E_0 = 12x^2 + 25x + 12$

Polynomial Mult: Divide & Conquer

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$E_0(x) = 3x + 4$

$D_1E_1 = 4x^2 + 11x + 6$

$D_0E_1 = 2x^2 + 5x + 2$

$D_1E_0 = 12x^2 + 25x + 12$

Polynomial Mult: Divide & Conquer

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$D_0E_0 = 6x^2 + 11x + 4$

Polynomial Mult: Divide & Conquer

$A(x) = 4x^3 + 3x^2 + 2x + 1$

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$AB = (4x^2 + 11x + 6)x^4 +$

Polynomial Mult: Divide & Conquer

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$$(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$$

Polynomial Mult: Divide & Conquer

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$D_0E_1 = 2x^2 + 5x + 2$

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$D_0E_0 = 6x^2 + 11x + 4$

$AB = (4x^2 + 11x + 6)x^4 +$

$(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$

$6x^2 + 11x + 4$

Polynomial Mult: Divide & Conquer

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$D_1E_1 = 4x^2 + 11x + 6$

$D_0E_1 = 2x^2 + 5x + 2$

$D_1E_0 = 12x^2 + 25x + 12$

$D_0E_0 = 6x^2 + 11x + 4$

$$AB = (4x^2 + 11x + 6)x^4 + (12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 + 6x^2 + 11x + 4$$

$$AB = (4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4)$$

Function Mult2(A, B, n, a_l , b_l)

$if\ n = 1: R = array[0..2n - 2]$

$R[0] = A[a_l] * B[b_l];\ return\ R$

$R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2}) R[0..n - 2]$
$= Mult2(A, B, \frac{n}{2}, a_l, b_l)$

$D_1E_0 = Mult2\left(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l\right)$
$D_0E_1 = Mult2(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

$R[\frac{n}{2} \ldots n + \frac{n}{2} - 2] += D_1E_0 + D_0E_1$

$return\ R$

Level		#	Work
0	n	1	kn
1	n/2 | n/2 | n/2 | n/2	4	$4k\frac{n}{2} = k2n$
⋮	⋮	⋮	⋮
i	… $n/2^i$ …	4^i	$4^i k\frac{n}{2^i} = k2^i n$
⋮	⋮	⋮	⋮
$\log_2 n$	1 … 1	$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

$$Total = \sum_{i=0}^{\log_2 n} 4^i k \frac{n}{2^i} = \theta(n^2)$$

Faster Divide – and – Conquer

Karatsuba Approach

Karatsuba Approach

$$A(x) = a_1x + a_0$$
$$B(x) = b_1x + b_0$$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

Need 4 Multiplications

Rewrite as:

$$C(x) = a_1b_1x^2 + \big((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0\big)x + a_0b_0$$

Need 3 Multiplications

Karatsuba Example

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

Karatsuba Example

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

$D_1(x) = 4x + 3$

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$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

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Karatsuba Example

$A(x) = 4x^3 + 3x^2 + 2x + 1$

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$D_1E_1 = 4x^2 + 11x + 6$

$D_0E_0 = 6x^2 + 11x + 4$

$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6) = 24x^2 + 52x + 24$

Karatsuba Example

$A(x) = 4x^3 + 3x^2 + 2x + 1$

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$\quad (24x^2 + 52x + 24 \qquad)x^2 +$

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$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6) = 24x^2 + 52x + 24$

$$AB = (4x^2 + 11x + 6)x^4 + \left(24x^2 + 52x + 24 - (4x^2 + 11x + 6) - (6x^2 + 11x + 4)\right)x^2 +$$

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$A(x) = 4x^3 + 3x^2 + 2x + 1$

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$D_1 E_1 = 4x^2 + 11x + 6$ $\quad\quad$ $D_0 E_0 = 6x^2 + 11x + 4$

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$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6) = 24x^2 + 52x + 24$

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$AB = (4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4)$

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1	n/2 | n/2 | n/2	3	$3k\frac{n}{2} = k\frac{3}{2}n$
⋮	⋮	⋮	⋮
i	… $n/2^i$ …	3^i	$3^i k\frac{n}{2^i} = k(\frac{3}{2})^i n$
⋮	⋮	⋮	⋮
$\log_2 n$	1 … 1	$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

$$Total = \sum_{i=0}^{\log_2 n} 3^i k\frac{n}{2^i} = \theta\left(n^{\log_2 n}\right) = \theta(n^{1.58})$$