Design and Analysis of Algorithms

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Greedy Algorithms

Grouping Children

Imran Ihsan
Assistant Professor, Department of Computer Science
Air University, Islamabad, Pakistan
www.imranihsan.com

The Problem

Many children came to a celebration.

Organize them into the minimum possible number of groups such that the age of any two children in the same group differ by at most one year.

Naïve Algorithm

```
MinGroups(C)
    m ← len(C)
    for each partition into groups
    C = G1 ∪ G2 ∪ · · · ∪ Gk:
        good ← true
        for i from 1 to k:
            if max(Gi) – min(Gi) > 1:
                good ← false
        if good:
            m ← min(m, k)
    return m
```

Running Time

The number of operations in `MinGroups(C)`

is at least 2^n, where `n` is the number of children in `C`.

Proof

Consider just partitions in two groups

$C = G_1 \cup G_2$

For each $G_1 \subset C$, $G_2 = C \setminus G_1$

Size of C is n

Each item can be included or excluded from G_1

There are 2^n different G_1

Thus, at least 2^n operations

Asymptotics

Naïve algorithm works in time $\Omega(2^n)$

For $n = 50$ it is at least

$$2^{50} = 1125899906842624$$

operations!

We will improve this significantly

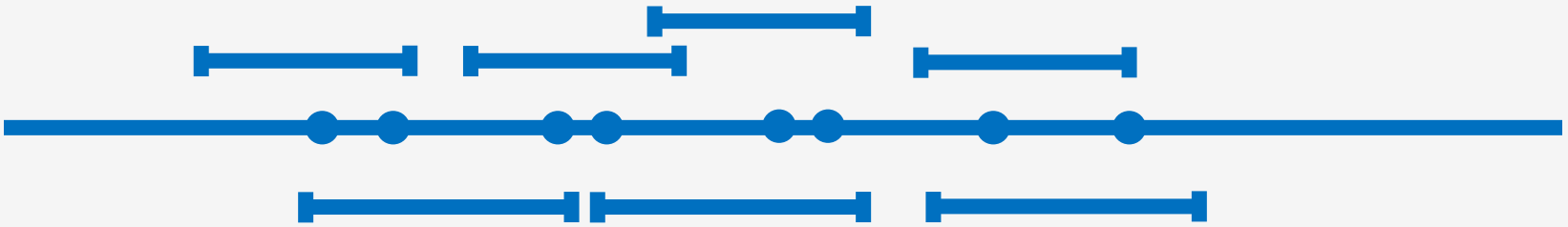
Efficient Algorithm

Covering Points by Segments

Input: A set of n points $x_1, \ldots, x_n \in R$.

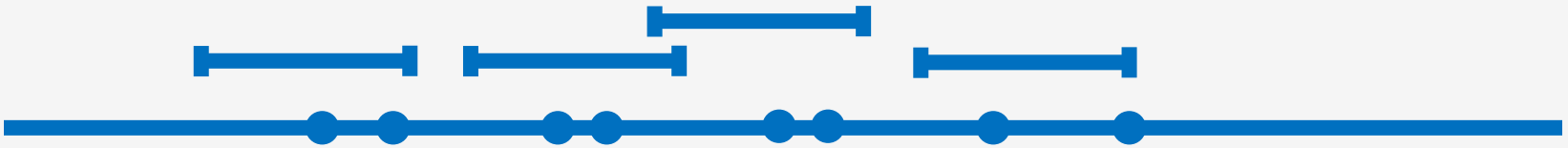
Output: The minimum number of segments of unit length needed to cover all the points.

Example



Safe Move

cover the leftmost point with a unit segment which starts in this point.



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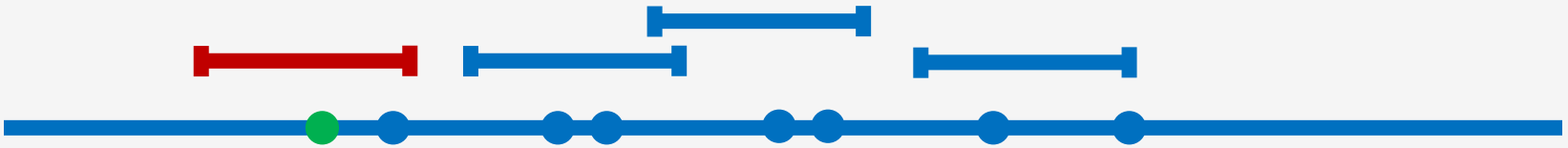

Safe Move

cover the leftmost point with a unit segment which starts in this point.

Efficient Algorithm

Assume $x_1 \leq x_2 \leq \ldots \leq x_n$

```
PointsCoverSorted(x_1, . . . , x_n)
    R ← {}, i ← 1
    while i ≤ n:
        [ℓ, r ] ← [x_i , x_i + 1]
        R ← R ∪ {[ℓ, r ]}
        i ← i + 1
        while i ≤ n and x_i ≤ r:
            i ← i + 1
    return R
```

Lemma

The running time of PointsCoverSorted is O(n).

Proof

i changes from 1 to n

For each i , at most 1 new segment

Overall, running time is O(n)

Total Running Time

PointsCoverSorted works in O(n) time

Sort $\{x_1, x_2, . . . , x_n\}$, then call PointsCoverSorted

Soon you'll learn to sort in O(n log n)

Sort + PointsCoverSorted is O(n log n)

Asymptotic

Straightforward solution is $\Omega(2^n)$

Very long for $n = 50$

Sort + greedy is $O(n \log n)$

Fast for $n = 10\ 000\ 000$

Huge improvement!

Conclusion

Straightforward solution is exponential

Important to reformulate the problem in mathematical terms

Safe move is to cover leftmost point

Sort in `O(n log n)` + greedy in `O(n)`